5. Time-Domain Analysis of Discrete-Time Signals and Systems

5.1. Impulse Sequence (1.4.1)

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5.3. Discrete-Time Impulse Response (2.1)

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5.1. Impulse Sequence

5.1.1. Impulse Sequence

The impulse sequence is defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases} . \tag{5.1}$$

The impulse sequence is illustrated in figure 5.1.

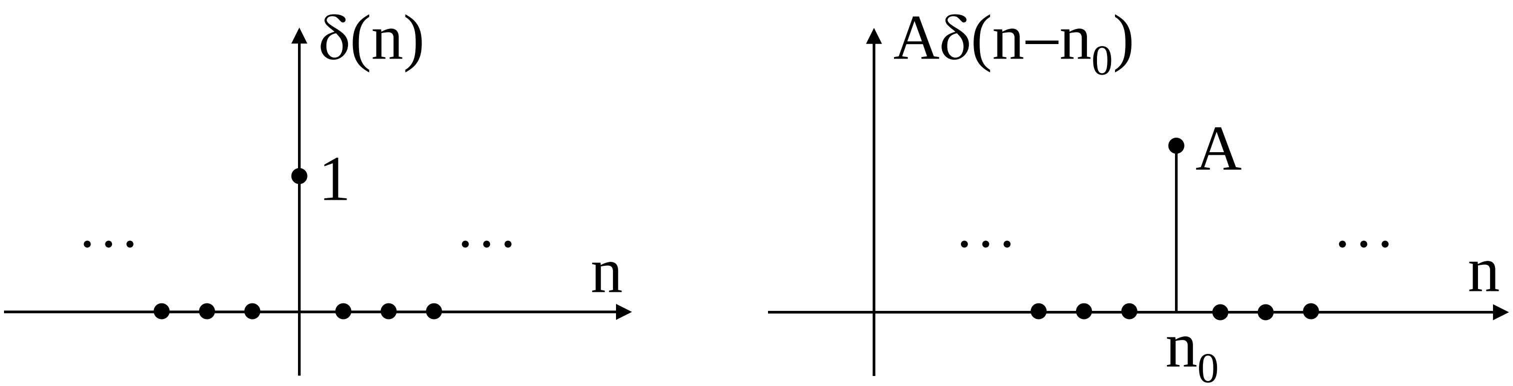

Figure 5.1. Impulse Sequence.

$\delta(n)$ has a sampling property, i.e.,

$$x(n)\delta(n-n_0)=x(n_0)\delta(n-n_0), \tag{5.2}$$

where n_0 is an arbitrary integer.

5.1.2. Step Sequence

The step sequence is defined as

$$u(n)=\begin{cases}0, & n<0\\ 1, & n\geq 0\end{cases}. \tag{5.3}$$

The step sequence is illustrated in figure 5.2.

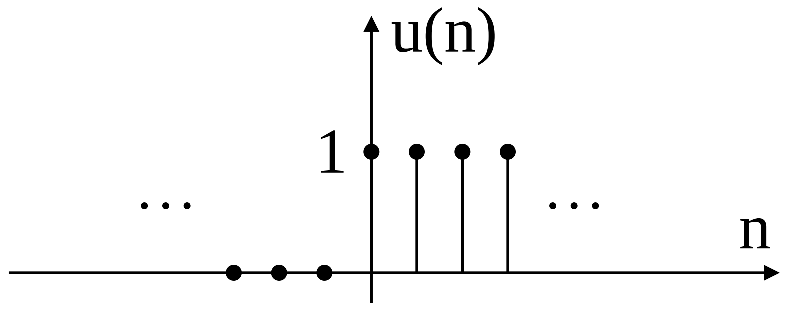

Figure 5.2. Step Sequence.

$u(n)$ can be expressed as the running sum of $\delta(n)$, i.e.,

$$u(n) = \sum_{m=-\infty}^{n} \delta(m). \tag{5.4}$$

$\delta(n)$ can be expressed as the difference of $u(n)$, i.e.,

$$\delta(n)=u(n)-u(n-1). \tag{5.5}$$

5.2. Convolution Sum

The convolution sum of $x_1(n)$ and $x_2(n)$ is defined as

$$x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m). \tag{5.6}$$

Note that the summation is carried out with respect to an introduced variable, m, and the final result is a function of n.

The convolution sum satisfies the commutative property

$$x_1(n)*x_2(n)=x_2(n)*x_1(n), \tag{5.7}$$

the associative property

$$[x_1(n)*x_2(n)]*x_3(n)=x_1(n)*[x_2(n)*x_3(n)], \tag{5.8}$$

and the distributive property

$$x_1(n)*[x_2(n)+x_3(n)]=x_1(n)*x_2(n)+ x_1(n)*x_3(n). \tag{5.9}$$

The convolution sum can be calculated in the following steps.

(1) Reflect $x_2(m)$ about the origin to obtain $x_2(-m)$.

(2) Shift $x_2(-m)$ by n to obtain $x_2(n-m)$.

(3) Calculate the convolution sum at n.

Steps (2) and (3) often need to be carried out in different ways for different intervals of n.

Example. Find $x_1(n)*x_2(n)$, where

(1) $x_1(n)=\begin{cases}0.5, & n=0\\ 2, & n=1\\ 0, & \text{otherwise}\end{cases}$ and $x_2(n)=\begin{cases}1, & 0\le n\le 2\\ 0, & \text{otherwise}\end{cases}$.

(2) $x_1(n)=a^n u(n)$ and $x_2(n)=u(n)$.

(3) $x_1(n)=\begin{cases}1, & 0\le n\le 4\\ 0, & \text{otherwise}\end{cases}$ and $x_2(n)=\begin{cases}a^n, & 0\le n\le 6\\ 0, & \text{otherwise}\end{cases}$.

(4) $x_1(n)=2^n u(-n)$ and $x_2(n)=u(n)$.

5.3. Discrete-Time Impulse Response

5.3.1. Definition of Discrete-Time Impulse Response

A linear time-invariant discrete-time system can be described by the discrete-time impulse response, which is defined as the response of the system to the impulse sequence.

A linear time-invariant discrete-time system can also be described by the discrete-time step response. It is defined as the response of the system to the step sequence.

5.3.2. I/O Relation by Discrete-Time Impulse Response

The I/O relation of a linear time-invariant discrete-time system can be expressed by its impulse response. Assume that x(n) and h(n) are the input and the impulse response of a linear time-invariant discrete-time system, respectively. Then, the output of the system is

$$y(n)=x(n)*h(n). \tag{5.10}$$

Proof. Since

$$x(n)=\sum_{m=-\infty}^{\infty}x(m)\delta(n-m), \tag{5.11}$$

the output of the system can be expressed as

$$y(n) = T[x(n)] = T\left[\sum_{m=-\infty}^{\infty} x(m)\delta(n-m)\right]. \qquad (5.12)$$

Since the system is linear, then

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)T[\delta(n-m)]. \qquad (5.13)$$

Since T[δ(n)]=h(n) and the system is time-invariant, then

$$T[\delta(n-m)]=h(n-m). \qquad (5.14)$$

Substituting (5.14) into (5.13), one obtains (5.10).

5.4. Classification of a Linear Time-Invariant Discrete-Time System by its Impulse Response

5.4.1. Memoryless Systems versus Systems with Memory

Assume that h(n) is the impulse response of a linear time-invariant discrete-time system. The system is memoryless if and only if

$$h(n)=0 \text{ for } n\neq 0. \tag{5.15}$$

5.4.2. Causal Systems versus Noncausal Systems

Assume that h(n) is the impulse response of a linear time-invariant discrete-time system. The system is causal if and only if

$$h(n)=0 \text{ for } n<0. \tag{5.16}$$

5.4.3. Stable Systems versus Unstable Systems

Assume that h(n) is the impulse response of a linear time-invariant discrete-time system. The system is stable if and only if h(n) is absolutely summable. We say that h(n) is absolutely summable when there exists a finite constant B such that

$$\sum_{n=-\infty}^{\infty} |h(n)| \leq B. \tag{5.17}$$

Proof. Consider the sufficiency first. Let x(n) be bounded, i.e.,

$$|x(n)| \le A, \tag{5.18}$$

where A is a finite constant. Then,

$$|y(n)| = \left| \sum_{m=-\infty}^{\infty} h(m)x(n-m) \right| \le \sum_{m=-\infty}^{\infty} |h(m)||x(n-m)|$$

$$\le A \sum_{m=-\infty}^{\infty} |h(m)| \le AB. \tag{5.19}$$

That is, y(n) is also bounded. Thus, the system is stable. Consider the necessity next. For the input

$$x(n) = \begin{cases} h^{*}(-n)/|h(-n)|, & h(-n) \ne 0 \\ 0, & h(-n) = 0 \end{cases}, \tag{5.20}$$

the output of the system at n=0 is

$$y(0) = \sum_{m=-\infty}^{\infty} x(m)h(-m) = \sum_{h(-m)\ne 0} x(m)h(-m)$$

$$= \sum_{h(-m)\neq 0} |h(-m)| = \sum_{m=-\infty}^{\infty} |h(-m)| = \sum_{n=-\infty}^{\infty} |h(n)|. \qquad (5.21)$$

The system is assumed to be stable. Then, since x(n) is bounded, y(n) is also bounded. Thus, there exists a finite constant B such that

$$\sum_{n=-\infty}^{\infty} |h(n)| \leq B. \qquad (5.22)$$

Example. Determine whether the following systems are stable:

(1) $h(n)=\delta(n-n_0)$.

(2) $h(n)=u(n)$

(3) $h(n)=0.5^n u(n)$.

(4) $h(n)=2^n u(n)$.

(5) $h(n)=0.5^n u(-n-1)$.

5.4.4. Invertible Systems versus Noninvertible Systems

We assume that two linear time-invariant discrete-time systems A and B have the impulse responses g(n) and h(n), respectively. A and B are mutually inverse if and only if

$$g(n)*h(n)=\delta(n). \tag{5.23}$$

(5.23) can be used to construct the inverse of a given system.

5.5. Linear Constant-Coefficient Difference Equations

A discrete-time system may be characterized by a linear constant-coefficient difference equation. However, it should be mentioned that only a linear constant-coefficient difference equation cannot specify a discrete-time system uniquely. Other conditions, such as some output samples under a given input or the statements about linearity, time-invariance, causality and stability, are also required.

A discrete-time system is often characterized by a linear constant-coefficient difference equation, a right-sided input and some initial conditions. We will focus on these cases.

5.5.1. Homogeneous Solution and Particular Solution

A linear constant-coefficient difference equation can be solved in the following steps.

(1) Find the homogeneous solution. α, a characteristic value of order K, corresponds to K terms in the homogeneous solution, i.e.,

$$y_h(n) = \ldots + \sum_{k=0}^{K-1} A_k n^k \alpha^n + \ldots, \tag{5.24}$$

where A_k is the coefficient to be determined.

(2) Find the particular solution. Let the free term be $P(n)\beta^n$, where $P(n)$ is a polynomial, and β is a characteristic value of order L. Then, the particular solution will have the form

$$y_p(n)=n^L Q(n)\beta^n, \tag{5.25}$$

where Q(n) is a polynomial with the same order as P(n). Substituting (5.25) into the linear constant-coefficient difference equation, one can determine the coefficients in Q(n). In addition, it should be noted that the particular solution is linear with respect to the free term.

(3) Combine the homogeneous solution and the particular solution to form the complete solution. Then determine the coefficients in the homogeneous solution.

The homogeneous solution is also called the natural response. The particular solution is also called the forced response.

Example. A discrete-time system is given by

$$y(n)-0.5y(n-1)=x(n), \tag{5.26}$$

where $x(n)=0.25^n u(n)$ and $y(-1)=2$. Find y(n).

First consider y(n) for n<0. When n<0, (5.26) becomes

$$y(n)-0.5y(n-1)=0. \tag{5.27}$$

The characteristic equation is

$$\lambda-0.5=0. \tag{5.28}$$

The characteristic value, i.e., the solution to (5.28), is $\lambda=0.5$. Thus, the homogeneous solution has the form

$$y_h(n)=A0.5^n. \tag{5.29}$$

Here, the homogeneous solution is also the complete solution. Thus,

$$y(n)=A0.5^n. \tag{5.30}$$

Using $y(-1)=2$, we obtain $A=1$. Thus,

$$y(n)=0.5^n. \tag{5.31}$$

Then consider $y(n)$ for $n\geq0$. When $n\geq0$, (5.26) becomes

$$y(n)-0.5y(n-1)=0.25^n. \tag{5.32}$$

The homogeneous solution has the form

$$y_h(n)=A0.5^n. \tag{5.33}$$

The particular solution has the form

$$y_p(n)=B0.25^n. \tag{5.34}$$

Substituting (5.34) into (5.32), we obtain B=–1. Thus,

$$y_p(n)=-0.25^n. \tag{5.35}$$

Adding (5.33) and (5.35), we obtain the complete solution, i.e.,

$$y(n)=A0.5^n-0.25^n. \tag{5.36}$$

Letting n=0 in (5.26), we obtain y(0)=2. Applying this condition to (5.36), we obtain A=3. Thus,

$$y(n)=3\cdot 0.5^n-0.25^n. \tag{5.37}$$

Finally, combining (5.31) and (5.37), we obtain

$$y(n)=0.5^n u(-n-1)+(3\cdot 0.5^n-0.25^n)u(n). \tag{5.38}$$

5.5.2. Zero-Input Response and Zero-State Response

The solution to a linear constant-coefficient difference equation can be decomposed into the inherent response and the response due to the input. The former is called the zero-input response because it equals the response when the input is zero. The latter is called the zero-state response because it equals the response when the state (the inherent response) is zero.

Example. A causal discrete-time system is given by

$$y(n)-0.5y(n-1)=x(n), \tag{5.39}$$

where $x(n)=0.25^n u(n)$ and $y(-1)=2$. Find the zero-input response, the zero-state response and the complete response.

The input is right-sided. Since the system is causal, the zero-state response is also right-sided. This means that $y(-1)$ has nothing to do the zero-state response and is only related to the zero-input response. Thus, the zero-input response is the solution to

$$y_{zi}(n)-0.5y_{zi}(n-1)=0, \tag{5.40}$$

where $y_{zi}(-1)=2$, and the zero-state response is the solution to

$$y_{zs}(n)-0.5y_{zs}(n-1)= 0.25^n u(n), \tag{5.41}$$

where $y_{zs}(-1)=0$. Using the method in section 5.5.1, we obtain the zero-input response

$$y_{zi}(n)=0.5^n \tag{5.42}$$

and the zero-state response

$$y_{zs}(n)=(2\cdot 0.5^n-0.25^n)u(n). \tag{5.43}$$

The complete response is the sum of the zero-input response and the zero-state response, i.e.,

$$y(n)=0.5^n+(2\cdot 0.5^n-0.25^n)u(n). \tag{5.44}$$

Figure 5.3 is used to clarify the homogeneous solution (the natural response), the particular solution (the forced response), the zero-input response, and the zero-state response in this example.

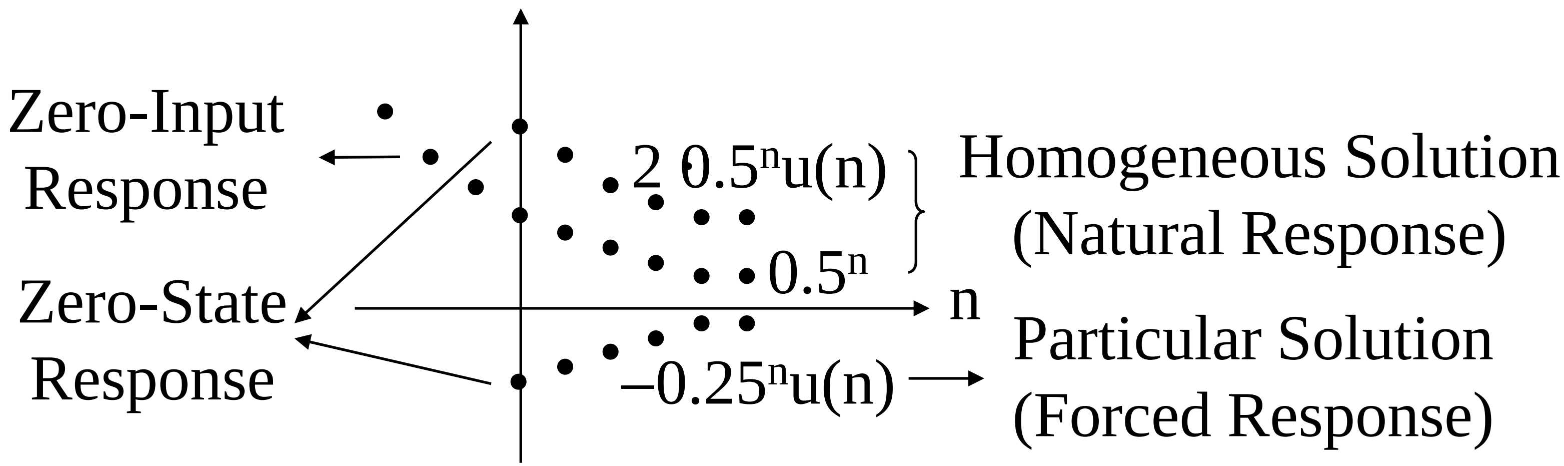

Figure 5.3. Clarification of Several Concepts.

The zero-state response is linear and time-invariant with respect to the input. This means that the relation between the input and the zero-state response can be characterized by an impulse response. The impulse response is the zero-state response to the impulse sequence. The zero-state response equals the convolution sum of the input and the impulse response.

Example. A causal discrete-time system is given by

$$y(n)-0.5y(n-1)=x(n). \tag{5.45}$$

Find the impulse response h(n).

The zero-state response satisfies

$$y_{zs}(n)-0.5y_{zs}(n-1)=x(n). \tag{5.46}$$

When $x(n)=\delta(n)$, $y_{zs}(n)=h(n)$. Thus,

$$h(n)-0.5h(n-1)=\delta(n). \tag{5.47}$$

Since the system is causal,

$$h(n)=0 \text{ for } n<0. \tag{5.48}$$

Letting n=0 in (5.47), we obtain

$$h(0)=1. \tag{5.49}$$

When n>0, h(n) is the solution to

$$h(n)-0.5h(n-1)=0, \tag{5.50}$$

where h(0)=1. Using the method in section 5.5.1, we obtain

$$h(n)=A0.5^n. \tag{5.51}$$

Letting n=1 in (5.50), we obtain h(1)=0.5. Applying this condition to (5.51), we obtain A=1. Thus,

$$h(n)=0.5^n. \tag{5.52}$$

Combining (5.48), (5.49) and (5.52), we obtain

$$h(n)=0.5^n u(n). \tag{5.53}$$