7. Sampling and Reconstruction

7.1. Sampling (7.1)
7.2. Reconstruction (7.2)
7.3. Sampling Theorem (7.1, 7.3)
7.1. Sampling

7.1.1. Time-Domain Expression of Sampling

A continuous-time signal is converted into a discrete-time signal by sampling. Sampling can be periodic or not. We consider periodic sampling only. It is expressed as

\[ x(n) = x_c(t) \big|_{t=nT} = x_c(nT). \]  \hspace{1cm} (7.1)

\( x_c(t) \) is a continuous-time signal. \( x(n) \) is the corresponding discrete-time signal. \( T \) is the sampling interval.

7.1.2. Frequency-Domain Expression of Sampling

Let \( x_c(t) \) be a continuous-time signal, \( X_c(\Omega) \) be the continuous-time Fourier transform of \( x_c(t) \), \( x(n) \) be a discrete-time signal, and \( X(\omega) \) be the discrete-time Fourier transform of \( x(n) \). If \( x(n) = x_c(nT) \), then
\[ X(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi m}{T}\right). \] (7.2)

Letting \( \omega = \Omega T \) in (7.2), we obtain

\[ X(\Omega T) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\Omega - \frac{2\pi m}{T}\right). \] (7.3)
$X(\Omega T)$ is the discrete-time Fourier transform of $x(n)$ expressed in $\Omega$. (7.3) shows that $X(\Omega T)$ equals $X_c(\Omega)$ extended with period $2\pi/T$ and divided by $T$ (figure 7.1).

(7.2) is proved as follows.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \exp(-j\omega n)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT) \exp(-j\omega n)$$

$$= \sum_{n=-\infty}^{\infty} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\Omega) \exp(j\Omega T n) d\Omega \right] \exp(-j\omega n)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\Omega) \left[ \sum_{n=-\infty}^{\infty} \exp(j\Omega T n) \exp(-j\omega n) \right] d\Omega$$

$$= \int_{-\infty}^{\infty} X_c(\Omega) \left[ \sum_{m=-\infty}^{\infty} \delta(\omega - \Omega T - 2\pi m) \right] d\Omega$$
\[ = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(\Omega)\delta(\omega - \Omega T - 2\pi m) d\Omega \]

\[ = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\frac{\omega - 2\pi m}{T}\right). \quad (7.4) \]

7.2. Reconstruction

A discrete-time signal is converted into a continuous-time signal by reconstruction.

This section uses the following notation:

- \( x_c(t) \) — a continuous-time signal,
- \( \Omega_0 \) — the central frequency of \( x_c(t) \),
- \( x(n) \) — the sequence obtained by sampling \( x_c(t) \),
- \( T \) — the sampling interval,
X(ω) — the discrete-time Fourier transform of x(n),
x′(t) — the continuous-time signal reconstructed from x(n),
X′(Ω) — the continuous-time Fourier transform of x′(t).

7.2.1. Frequency-Domain Expression of Reconstruction

The frequency-domain expression of reconstruction is

\[
X′(Ω) = \begin{cases} 
TX(Ω)T, & Ω_0 - \pi / T ≤ Ω < Ω_0 + \pi / T \\
0, & \text{otherwise} 
\end{cases}
\]  

(7.5)

(7.5) is obtained from the spectral relation in sampling.

Especially, when \( x_c(t) \) is real, \( Ω_0 = 0 \) and (7.5) becomes

\[
X′(Ω) = \begin{cases} 
TX(Ω)T, & -\pi / T ≤ Ω < \pi / T \\
0, & \text{otherwise} 
\end{cases}
\]  

(7.6)
7.2.2. Time-Domain Expression of Reconstruction

The time-domain expression of reconstruction is

$$x'_c(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T} \exp[j\Omega_0(t - nT)].$$  \hspace{1cm} (7.7)

Especially, when $x_c(t)$ is real, $\Omega_0=0$ and (7.7) becomes

$$x'_c(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}.$$  \hspace{1cm} (7.8)

(7.7) is proved as follows.

$$x'_c(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X'_c(\Omega) \exp(j\Omega t) d\Omega$$

$$= \frac{T}{2\pi} \int_{\Omega_0 - \pi/T}^{\Omega_0 + \pi/T} X(\Omega T) \exp(j\Omega t) d\Omega$$
\[
\begin{align*}
&= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \left[ \sum_{n=-\infty}^{\infty} x(n) \exp(-j\Omega Tn) \right] \exp(j\Omega t) d\Omega \\
&= \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} x(n) \int_{-\pi/T}^{\pi/T} \exp[j\Omega(t - nT)] d\Omega \\
&= \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T} \exp[j\Omega_0(t - nT)].
\end{align*}
\] (7.9)

7.3. Sampling Theorem

Let \( x_c(t) \) be a continuous-time signal with bandwidth \( W \) and \( x(n) \) be the discrete-time signal obtained by sampling \( x_c(t) \) with sampling interval \( T \). If

\[
2\pi/T > W,
\] (7.10)

\( x_c(t) \) can be reconstructed from \( x(n) \) without distortion (figure 7.2). Otherwise, the reconstruction may have distortion owing to aliasing (figure 7.3).
$2\pi/T$ is referred to as the sampling frequency, and $W$ is referred to as the Nyquist rate.

Especially, when $x_c(t)$ is real, the central frequency is 0, and (7.10) becomes

$$2\pi/T>2\Omega_u,$$  \hspace{1cm} (7.11)
where $\Omega_u$ is the upper limit of the band.

Example. $x_c(t)$ is a continuous-time signal with central frequency $\Omega_0$ and bandwidth $W$. $x(n)$ is the discrete-time signal obtained by sampling $x_c(t)$ with sampling interval $T$. $x'_c(t)$ is the continuous-time signal with central frequency $\Omega_0$ and bandwidth $W$. Figure 7.3. $2\pi/T \leq W$. 

![Diagram of continuous and discrete-time signals and their frequency representations.](image-url)
signal reconstructed from \( x(n) \). When \( 2\pi/T \leq W \), \( x'_c(t) \) may not equal \( x_c(t) \). \( x'_c(t) \) does not equal \( x_c(t) \) either when there is a wideband noise. However, the difference between \( x'_c(t) \) and \( x_c(t) \) can be decreased by filtering \( x_c(t) \) before it is sampled. The filter has frequency response

\[
H_c(\Omega) = \begin{cases} 
1, & \Omega_0 - \frac{\pi}{T} \leq \Omega < \Omega_0 + \frac{\pi}{T} \\
0, & \text{otherwise}
\end{cases}
\]  

(7.12)

Explain how it works.