APPLIED AND COMPUTATIONAL HARMONIC ANALYSIS **4,** 317–350 (1997)
ARTICLE NO. HA970216

Wavelets Generated by Vector Multiresolution Analysis[1]

Ruilin Long,† Wen Chen,[2] and Shenglan Yuan

Institute of Mathematics, Academia Sinica, Beijing 100080, China

Communicated by Albert Cohen

Received November 2, 1995; revised December 10, 1996

The paper presents a general approach to the construction of so-called biorthogonal vector-MRA and its related wavelets of $L^2(R^d)$. The presented algorithm is very close to the one in the classical case given by Cohen–Daubechies ($d = 1$) and Long–Chen ($d \geqslant 1$). Roughly speaking, to get a biorthogonal vector-MRA from a given couple $\{H_0(\xi); \tilde{H}_0(\xi)\}$ of trigonometric polynomial matrices satisfying $\Sigma_\nu H_0(\xi + \nu\pi)\tilde{H}_0^*(\xi + \nu\pi) = I_m$ (modulo some other natural mild conditions), it is needed only to check if both of the spectral radius of the transition operators P_{H_0}, and $P_{\tilde{H}_0}$ restricted on some suitable invariant space $\mathcal{P}_0$, are less than 1. © 1997 Academic Press

1. INTRODUCTION

During the past decade, the wavelet analysis based on the multiresolution analysis (MRA) with a single scaling function has undergone a flourishing development. There are many choices of wavelets constructed by various MRA, each possessing various combinations of desirable properties such as orthogonality, compact support, smoothness, symmetry, or high accuracy. However, some of these properties are mutually exclusive. For instance, there are no compactly supported orthogonal wavelets, other than the Haar wavelet, which can be symmetric or antisymmetric (see [12]). By the way, the accuracy and smoothness of the scaling function is tied to the number of coefficients in the dilation equation (see [13]). Therefore to obtain the high accuracy or smoothness wavelets implies enlarging the support size for the scaling function. This reduces the locality of wavelet representation and increases the computational

† Deceased.

[1] Supported by the National Sciences Foundation of China.

[2] Current address: Department of Information Network Sciences, Graduate School of Information Systems, University of Electro-Communications, Chofugaoka 1-5-1, Chofu 182, Tokyo, Japan. E-mail: wchen@net.is.uec.ac.jp.

1063-5203/97 $25.00
Copyright © 1997 by Academic Press
All rights of reproduction in any form reserved.

complexity of the wavelet transform. In order to improve the support size, or the symmetry of the involved wavelets, recently the so-called vector MRA(m) began to be introduced by many people. One advantage of the wavelets based on the vector MRA(m) allows the simultaneous inclusion of desirable properties. The contribution of Goodman and Lee [16], Hardin, Kessler, and Massopust [18], Geronimo, Hardin, and Massopust [15], and Donovan, Geronimo, Hardin, and Massopust [14] were among the first in this direction. The idea is to let a multiresolution analysis $\{V_j\}_{-\infty}^{\infty}$ of $L^2(R^d)$ be based on a vector function $(\varphi_1, \varphi_2, \ldots, \varphi_m) \in L^2(R^d, R^m)$ satisfying that $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}$ consists in a Riesz basis of V_0. Hence $\{2^{d/2}\varphi_1(2x-k), \ldots, 2^{d/2}\varphi_m(2x-k)\}$ is a Riesz basis of V_1. Since $V_0 \subset V_1$, there are $\{d_{i,j}(k)\}_k \in l^2(Z^d)$, $i, j = 1, \ldots, m$, such that (in $L^2(R^d)$ sense)

$$\varphi_i(x) = 2^d \sum_{j=1}^{m} \sum_k d_{i,j}(k)\varphi_j(2x-k), \quad i = 1, \ldots, m. \tag{1.1}$$

In terms of Fourier transform, (1.1) becomes

$$\hat{\varphi}_i(\xi) = \sum_{j=1}^{m} \sum_k d_{i,j}(k) e^{-ik(\xi/2)} \hat{\varphi}_j\left(\frac{\xi}{2}\right), \quad i = 1, \ldots, m, \tag{1.2}$$

where the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{R^d} f(x) e^{-ix\cdot\xi} dx \quad \forall f \in L^1 \cap L^2. \tag{1.3}$$

Equation (1.2) can be rewritten in matrix form

$$\hat{\varphi}(\xi) = H\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right), \tag{1.4}$$

where $H(\xi) = (h_{i,j}(\xi))$ is a $m \times m$-matrix with $h_{i,j}(\xi) = \sum_k d_{i,j}(k) e^{-ik\cdot\xi}$ in $L^2(T^d)$. $H(\xi)$ is called the filter function matrix, and $\varphi = (\varphi_1, \ldots, \varphi_m)^{\mathrm{t}}$ is called the scaling function vector (t denotes the transpose).

On the contrary, suppose that $H(\xi)$ is a $2\pi Z^d$-periodic function matrix such that

(A) $\prod_1^{\infty} H(2^{-j}\xi)$ converges for all $\xi \in R^d$.

(B) For a $x \in R^m$, $Q(\xi) = \prod_1^{\infty} H(2^{-j}\xi)x$ is in $L^2(R^d, R^m)$.

(C) $S = \{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_{k \in Zd}$ is a Riesz system in $L^2(R^d)$, where $\varphi = (\varphi_1, \ldots, \varphi_m)^{\mathrm{t}}$ is the inverse Fourier transform of Q.

Then $V_0 = \overline{\mathrm{span} S}$ generates a vector MRA(m) with filter function matrix $H(\xi)$, and scaling function vector φ. If S is an orthogonal system, the associated MRA(m) is said to be orthogonal. If H and $\tilde{H}$ are two filter matrix functions such that the corresponding systems S and $\tilde{S}$ are biorthogonal, then the two associated MRAs are said to be biorthogonal. This leads to the further problems:

(D) Find conditions for given matrix function H to be filter function matrix generating orthogonal MRA(m).

(E) Find conditions for given matrix function H and $\tilde{H}$ to be filter function matrices generating biorthogonal MRAs.

The previous method was introduced by Mallat [29] and Daubechies [12] for constructing orthogonal MRAs in the case $d = m = 1$. It has been generated by several authors. For instance, in the case $d = m = 1$, problem (D) was studied by Cohen [4], Cohen and Raugi [10], and Lawton [23]; problems (B) and (C) were studied by Hérve [20] and Villemoes [31]. In the case $d \geqslant 1$, $m = 1$, problems (B) and (E) were studied by Cohen and Daubechies [6] and Cohen, Dabechies, and Feauveau [7]; problems (D) and (E) were studied by Long and Chen [26]. In the case $d = 1$, $m \geqslant 1$, problems (A), (B), and (C) were studied by Hérve [21]; problems (A) and (D) were studied in Donovan, Geronimo, Hardin, and Massopust [14], Chui and Lian [3], Lawton, Lee, and Shen [24], and Strang and Strela [30]; Problem (E) was studied by Dahmen and Micchelli [11]. The existence, uniqueness, regularity, and stability of φ as the solution of Eq. (1.4) were studied by Cohen, Daubechies, and Plonka [9], Heil and Colella [19], Cohen, Dyn, and Levin [8], and Lawton, Lee, and Shen [24] in the case $m \geqslant 1$. Following the previous works, especially those of Cohen and Daubechies [5], Hérve [21], and Long and Chen [26], we want to study (A), (D), and (E) in the general case $d \geqslant 1$ and $m \geqslant 1$, by using, first, the classical methods introduced by Cohen and Daubechies [5] and Long and Chen [26] and, second, arguments of uniform integrability which allow to simplify proofs; i.e., we want to find some conditions (necessary or sufficient) imposed on $H(\xi)$ such that $\prod_1^\infty H(2^{-j}\xi)$ converges and $\hat{\varphi}(\xi)$ can be defined by $\prod_1^\infty H(2^{-j}\xi)$ such that $\{V_j\}_{-\infty}^{\infty}$ is an orthogonal MRA(m) with

$$V_0 = \overline{\operatorname{span}}(\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k), \quad V_j = 2^j V_0. \tag{1.6}$$

We want also to study the wavelets generated by MRA(m), and the biorthogonal versions of the results obtained in the orthogonal case. As a result, for the filter function matrix $H(\xi)$ general enough (more general than those in [21]) in place of the filter function $m_0(\xi)$, we obtain almost all of the results of Long and Chen [26].

Section 2 will be devoted to the convergence of the infinite product $\prod_1^\infty H(2^{-j}\xi)$; Section 3 will be devoted to the characterization of the orthonormality of $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k$ for some kind of $H(\xi)$; Section 4 will discuss the wavelets generated by MRA(m); Section 5 will be the biorthogonal versions; and Section 6 will be the algorithms and two examples to illustrate the algorithms.

2. CONVERGENCE OF $\prod_1^\infty H(2^{-j}\xi)$

Denote $\mathcal{M}_m$ the set of all complex $m \times m$-matrices, $\mathcal{M}_m^+$ the set of all positive definite matrices. A $\in \mathcal{M}_m^+$ means that

$$\langle Ax, x\rangle = x^*Ax > 0 \quad \forall(\text{column}) \text{ vector } x(\neq 0). \tag{2.1}$$

For $A \in \mathcal{M}_m$, $\|A\|$ denotes the operator norm of A defined by

$$\|A\| = \sup_{|x|=1} |Ax|, \tag{2.2}$$

with $|x|$ the vector length of x. Denote e_i the ith-coordinate (column) vector. Let $H(\xi)$ be $m \times m$-matrix of continuous complex functions defined on T^d, in symbols $H \in C(T^d, \mathcal{M}_m)$. In this section, we want to study some necessary conditions and sufficient conditions for the convergence of $\prod_1^\infty H(2^{-j}\xi)$.

THEOREM 2.1. *Suppose that* $\prod_1^\infty H(2^{-j}\xi)$ *converges at* $\xi = 0$. *Then there exists nonsingular* $M = (m_{i,j}) \in \mathcal{M}_m$ *such that*

$$H(0) = M^{-1}\begin{pmatrix} I_s & 0 \\ 0 & D \end{pmatrix} M, \tag{2.3}$$

where I_s *is the* $(s \times s)$*-identity matrix,* $0 \leqslant s \leqslant m$, *and* D *is a special Jordan type matrix, i.e.,* D *is of type*

$$D = \begin{pmatrix} \lambda_{s+1} & \mu_{s+1} & & \\ & \ddots & \ddots & \\ & & \ddots & \mu_{m-1} \\ & & & \lambda_m \end{pmatrix}, \quad \mu_i = 0, 1 \text{ for all } i, \tag{2.4}$$

with $|\lambda_i| < 1$ *for all* i.

Proof. As is well known, there is a nonsingular $M \in \mathcal{M}$ such that

$$H(0) = M^{-1}\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_r \end{pmatrix} M,$$

with

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix} \in \mathcal{M}_{t_i}, \quad 1 \leqslant i \leqslant r \leqslant m, \sum_1^r t_i = m.$$

Since

$$H^n(0) = M^{-1}\begin{pmatrix} J_1^n & & \\ & \ddots & \\ & & J_r^n \end{pmatrix}M, \tag{2.5}$$

we see that $H^n(0)$ converges if and only if each J_i^n converges. But from

$$J_i^n = \begin{pmatrix} \lambda_i^n & n\lambda_i^n & + & \cdots & \cdots & * \\ & \lambda_i^n & & & \cdots & * \\ & & & & \ddots & \vdots \\ & & & & & \lambda_i^n \end{pmatrix}, \tag{2.6}$$

we see that when $t_i > 1$, then $|\lambda_i| < 1$, otherwise $n\lambda_i^{n-1}$ does not converge. When $t_i = 1$, we have $J_i^n = (\lambda_i^n)$. From the convergence we have $|\lambda_i| < 1$, or $\lambda_i = 1$. This completes the proof of the assertion of the theorem.

Now we want to show that the preceding necessary condition is almost sufficient. We assume a natural and mild condition which is needed even in the case $m = 1$.

THEOREM 2.2. *Let $H(\xi) \in C(T^d, \mathcal{M}_m)$ such that for some $\epsilon > 0$,*

$$\|H(\xi) - H(0)\| \leqslant c|\xi|^{\epsilon}. \tag{2.7}$$

Assume that (2.3) *holds with $s \geqslant 1$. Then on any compact set in R^d, $\{\prod_1^n H(2^{-j}\xi)\}_n$ converges uniformly to a continuous function matrix $\Pi_\infty(\xi)$. Furthermore, for $i > s$, $M\Pi_\infty(\xi)M^{-1}e_i = 0$.*

Proof. Denote $G(\xi) = MH(\xi)M^{-1}$. Then $\{\prod_1^n H(2^{-j}\xi)\}_n$ converges if and only if $\{\Pi_1^n G(2^{-j}\xi)\}_n$ converges. So without loss of generality we can assume

$$H(0) = \begin{pmatrix} I_s & \\ & D \end{pmatrix}.$$

Consider the compact set $D = [-N, N]^d$. It was shown by Cohen, Daubechies, and Plonka [9] that

$$\|\Pi_n(\xi)\| = \|\prod_1^n H(2^{-j}\xi)\| \leqslant c_{\delta,D}(1 + \delta)^n \quad \forall n \in Z_+ \ \forall \xi \in D.$$

Hence, for any $p, q \in Z_+$ with $q > p$ and any $\xi \in D$

$$\|\prod_{p+1}^{q} H(2^{-j}\xi)\| = \|\prod_1^{q-p} H(2^{-j}2^{-p}\xi)\| \leqslant c_{\delta,D}(1 + \delta)^{q-p}.$$

Since $H^k(0) \to \operatorname{diag}\{I_s, 0_{m-s}\}$, we can assume that $\|H^k(0)\| \leqslant B_H \leqslant O(1)$ for any $k \in Z_+$.

Denote the (i, k)-entry of $\prod_{j=1}^{n} H(2^{-j}\xi)$ by $h_{i,k}^{(n)}(\xi)$. We have

$$h_{i,k}^{(n)}(\xi) = \langle \prod_{j=1}^{n} H(2^{-j}\xi)e_k, e_i\rangle. \tag{2.10}$$

We rewrite

$$h_{i,k}^{(q)}(\xi) = \langle \prod_{j=p+1}^{q} H(2^{-j}\xi)e_k, (\prod_{j=1}^{p} H(2^{-j}\xi))^* e_i\rangle \tag{2.11}$$

and

$$\sum_l h_{i,l}^{(p)}(\xi)h_{l,k}^{(q-p)}(0) = \langle H^{q-p}(0)e_k, (\prod_{j=1}^{p} H(2^{-j}\xi))^* e_i\rangle. \tag{2.12}$$

Thus, we get

$$\begin{aligned}
&|h_{i,k}^{(q)}(\xi) - \sum_l h_{i,l}^{(p)}(\xi)h_{l,k}^{(q-p)}(0)| \\
&\quad = |\langle(\prod_{j=p+1}^{q} H(2^{-j}\xi) - H^{q-p}(0))e_k, (\prod_{j=1}^{p} H(2^{-j}\xi))^* e_i\rangle| \\
&\quad \leqslant \|(\prod_{j=1}^{p} H(2^{-j}\xi))^*\| \, \|\prod_{j=p+1}^{q} H(2^{-j}\xi) - H^{q-p}(0)\| \\
&\quad \leqslant c_{\delta,D}(1+\delta)^p\|\Delta\|,
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
\Delta &= \prod_{j=p+1}^{q} H(2^{-j}\xi) - H^{q-p}(0) \\
&= \prod_{j=p+1}^{q} H(2^{-j}\xi) - \prod_{j=p+1}^{q-1} H(2^{-j}\xi)H(0) \\
&\quad + \prod_{j=p+1}^{q-1} H(2^{-j}\xi)H(0) - \cdots + H(2^{-(p+1)}\xi)H^{q-p-1}(0) - H^{q-p}(0) \\
&= \sum_{l=0}^{q-p-1} \prod_{j=p+1}^{q-l} H(2^{-j}\xi)H^l(0) - \prod_{j=p+1}^{q-l-1} H(2^{-j}\xi)H^{l+1}(0) \\
&= \sum_{l=0}^{q-p-1} \prod_{j=p+1}^{q-l-1} H(2^{-j}\xi)(H(2^{-(q-l)}\xi) - H(0))H^l(0).
\end{aligned}$$

Since

$$\begin{aligned}
\|\Delta\| &\leqslant \sum_{l=0}^{q-p-1} c_{\delta,D}(1+\delta)^{q-p-l-2}\|H(2^{-(q-l)}\xi) - H(0)\| \, \|H^l(0)\| \\
&\leqslant \sum_{l=0}^{q-p-1} c_{\delta,D}(1+\delta)^{q-p-l-2}2^{-(q-l)\epsilon}N^{d\epsilon}B_H,
\end{aligned} \tag{2.14}$$

we derive

$$\begin{aligned}|h_{i,k}^{(q)}(\xi) - \sum_l h_{i,l}^{(p)}(\xi)h_{l,k}^{(q-p)}(0)| &\leqslant B_H N^{d\epsilon} c_{\delta,D}^2 \sum_{l=0}^{q-p-1} (1+\delta)^{q-l-2} 2^{-(q-l)\epsilon} \\ &= B_H N^{d\epsilon} c_{\delta,D}^2 (1+\delta)^{-2} \sum_{l=p+1}^{q} \left(\frac{1+\delta}{2^\epsilon}\right)^l. \qquad (2.15)\end{aligned}$$

Now we select a $\delta > 0$ such that $1 + \delta < 2^\epsilon$. Then $\{\sum_{l=0}^{q} ((1+\delta)/2^\epsilon)^l\}_q$ is a Cauchy sequence. When $k \leqslant s$, $h_{l,k}^{(q-p)}(0) = \delta_{l,k}$. Then (2.15) reads

$$|h_{i,k}^{(q)}(\xi) - h_{i,k}^{(p)}(\xi)| \leqslant B_H N^{d\epsilon} c_{\delta,D}^2 (1+\delta)^{-2} \sum_{l=p+1}^{q} \left(\frac{1+\delta}{2^\epsilon}\right)^l.$$

This means that $\{h_{i,k}^{(q)}(\xi)\}_q$ is a Cauchy sequence and, hence, converges. When $k > s$, let q tends to the infinity in (2.15) for given p large enough, we have that $|h_{i,k}^{(\infty)}(\xi)| \leqslant O(\sum_{l=p+1}^{\infty} ((1+\delta)/2^\epsilon)^l)$, since $\lim_{q\to\infty} h_{l,k}^{(q-p)}(0) = 0$. This means that $h_{i,k}^{(\infty)}(\xi) = 0$, for $k > s$ and all i. It is just the assertion: $M\Pi_\infty(\xi)M^{-1}e_i = 0$ for all $i > s$. The proof of the theorem is finished.

Remark. Hérve [21] obtained a similar result by assuming $H \sim \text{diag}\{1, \lambda_2, \ldots, \lambda_m\}$, but here we cannot use $\|H(0)\| = 1$ and $\|\prod_1^n H(2^{-j}\xi)\| \leqslant O(1)$ used by Hérve [21]. Furthermore, the result implies that we can construct a more general class of wavelet other than these with diagonal filter function matrix. The similar results were also found by Cohen, Daubechies, and Plonka [9] and Heil and Colella [19] with the different proofs.

3. ORTHOGONAL MRA(m)

The $H(\xi)$'s considered in this section are a little less general than those in Section 2. Assume that $H(\xi)$ satisfies the conditions in Section 2 with M in (2.3) being a unitary matrix, and s in (2.3) being 1. We want to show, in order to get an orthogonal MRA(m), what kinds of conditions (necessary or sufficient) should be satisfied.

At first, we have an obvious necessary condition: for any filter function matrix $H(\xi)$ of orthogonal MRA(m), we have

$$\sum_{\nu\in E_d} H(\xi+\nu\pi)H^*(\xi+\nu\pi) = I_m, \quad \text{a.e. } \xi, \qquad (3.1)$$

where $E_d = \{\text{all vertices of } [0,1]^d\}$. It follows from

$$\Phi(\xi) = \sum_{\alpha\in Z^d} \hat{\varphi}(\xi+2\pi\alpha)\hat{\varphi}^*(\xi+2\pi\alpha) = I_m, \quad \text{a.e. } \xi, \qquad (3.2)$$

which is an equivalent condition of the orthonormality of $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k$. In fact, writing $\alpha = 2\beta + \nu$, $\alpha, \beta \in Z^d$, $\nu \in E_d$, we get

$$I_m = \sum_\alpha H\left(\frac{\xi}{2} + \pi\alpha\right)\hat{\varphi}\left(\frac{\xi}{2} + \pi\alpha\right)\hat{\varphi}^*\left(\frac{\xi}{2} + \pi\alpha\right)H^*\left(\frac{\xi}{2} + \pi\alpha\right)$$

$$= \sum_\nu H\left(\frac{\xi}{2} + \nu\pi\right) \sum_\beta \hat{\varphi}\left(\frac{\xi}{2} + \nu\pi + 2\pi\beta\right)\hat{\varphi}^*\left(\frac{\xi}{2} + \nu\pi + 2\pi\beta\right)H^*\left(\frac{\xi}{2} + \nu\pi\right).$$

The fact that the orthonormality of $\{\varphi_1(x - k), \ldots, \varphi_m(x - k)\}_k$ is equivalent to (3.2) is well known. To be complete, we state a more general proposition as follows.

PROPOSITION 3.1. *Let $\{\varphi_i\}_1^m \subset L^2(R^d)$. Then $\{\varphi_1(x - k), \ldots, \varphi_m(x - k)\}_k$ has the upper, lower Riesz bounds B, A, if and only if*

$$AI_m \leqslant \Phi(\xi) \leqslant BI_m, \quad \text{a.e.} \tag{3.3}$$

Remark. The proposition is the natural and obvious extension of a well-known result; see, for example, [17, 2, 26, 25]. Notice that when $[\hat{\varphi}_i, \hat{\varphi}_i] \in L^\infty(T^d)$, the condition $\Phi(\xi) = ([\hat{\varphi}_i, \hat{\varphi}_j])_{i,j} \approx \mathrm{I}_m$ a.e. ξ is equivalent to a simple assertion $\det \Phi(\xi) \approx 1$ a.e. ξ, as shown by de Boor, DeVore, and Ron [2] and Long [25], where $[\cdot, \cdot]$ is the bracket product defined by

$$[f, g](\xi) = \sum_\alpha f(\xi + 2\pi\alpha)\overline{g}(\xi + 2\pi\alpha). \tag{3.7}$$

In what follows, the transition operators introduced and studied in wavelet theory by many people, such as Conze and Raugi [10], Lawton [23], Villemoes [31], Cohen and Daubechies [5], and Long and Chen [26], play a very important role. Now we define it. Let $H(\xi) \in C(T^d, \mathcal{M}_m)$ satisfying (3.1). Assume that there exists a unitary matrix M such that

$$H(0) = M^{-1}\begin{pmatrix} 1 & \\ & D \end{pmatrix}M, \tag{3.8}$$

with

$$D = \begin{pmatrix} \lambda_2 & \mu_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \mu_{m-1} & \\ & & & \lambda_m & \end{pmatrix}, \quad |\lambda_i| < 1, \mu_i = 0, 1 \quad \text{for all } i.$$

Define

$$P_H f(\xi) = \sum_{\nu \in E_d} MH\left(\frac{\xi}{2} + \nu\pi\right)M^{-1}f\left(\frac{\xi}{2} + \nu\pi\right)MH^*\left(\frac{\xi}{2} + \nu\pi\right)M^{-1}. \tag{3.9}$$

Obviously, P_H is an operator mapping measurable $2\pi Z^d$-periodic function matrices or continuous $2\pi Z^d$-periodic function matrices to matrices of the same kinds. And when

$H(\xi)$ is a trigonometric polynomial matrix, defining $\mathcal{P}$ to be a space of trigonometric polynomial matrices of N-degree (N depends only on the degree of H and will be specified in Section 6), then P_H maps $\mathcal{P}$ into $\mathcal{P}$. This is the same situation as in the classical case; see, for example, [21, 26]. Now the new feature is what is the right definition of some special invariant subspaces C_0 and $\mathcal{P}_0$. Notice that $G(\xi) = MH(\xi)M^{-1}$ satisfies

$$\sum_{\nu} G(\xi + \nu\pi)G^*(\xi + \nu\pi) = \sum_{\nu} MH(\xi + \nu\pi)H^*(\xi + \nu\pi)M^{-1} = I_m \quad (3.10)$$

and

$$G^*(\nu\pi)e_1 = \delta_{0,\nu}e_1, \quad \nu \in E_d. \quad (3.11)$$

Equation (3.11) follows from $G^*(0)e_1 = e_1$ and

$$1 = e_1^*e_1 = |G^*(0)e_1|^2 + \sum_{\nu\neq 0} |G^*(\nu\pi)e_1|^2 = 1 + 0.$$

Now define

$$C_0(T^d, \mathcal{M}_m) = \{ f(\xi) \in C(T^d, \mathcal{M}_m): (f(0))_{1,1} = 0\} \quad (3.12)$$

and

$$\mathcal{P}_0(T^d, \mathcal{M}_m) = \{ f(\xi) \in \mathcal{P}(T^d, \mathcal{M}_m): (f(0))_{1,1} = 0\}, \quad (3.13)$$

where $(f(\xi))_{i,j}$ denotes the (i, j)-entry of $f(\xi)$. Then we have

PROPOSITION 3.2. *Let $H(\xi) \in C(T^d, \mathcal{M}_m)$ be such that* (3.11) *holds, with $G(\xi) = MH(\xi)M^{-1}$. Then both C_0 and $\mathcal{P}_0$ are invariant subspaces of P_H.*

Proof. We only prove that when $f(\xi) \in C(T^d, \mathcal{M}_m)$ satisfies $(f(0))_{1.1} = 0$; we also have $(Tf(0))_{1.1} = 0$. This follows from

$$\begin{aligned}(Tf(0))_{1,1} &= e_1^* \sum_{\nu} MH(\nu\pi)M^{-1}f(\nu\pi)MH^*(\nu\pi)M^{-1}e_1 \\ &= e_1^*f(0)e_1 + \sum_{\nu\neq 0} 0^*f(\nu\pi)0 = 0.\end{aligned}$$

The proof is finished.

Some significant properties of the transition operators are formulated in the following proposition, which in fact is the obvious extension of the previous case (see [26, 21]). Here we only state the proposition without proof.

PROPOSITION 3.3. *Let $H(\xi) \in C(T^d, \mathcal{M}_m)$ be such that* (2.7), (3.1), *and* (3.8) *hold. Then P_H has two invariant matrices, i.e., I_m and $M\Phi(\xi)M^{-1}$, where $\Phi(\xi) = \Sigma_\alpha\, \hat{\varphi}(\xi +$*

$2\pi\alpha)\hat{\varphi}^*(\xi + 2\pi\alpha)$ *with* $\hat{\varphi}(\xi) = \prod_1^\infty H(2^{-j}\xi)x$, *for any* x. *Moreover, for any measurable* $2\pi Z^d$*-periodic function matrices* $f(\xi)$, $g(\xi)$ *we have*

$$P_H^n f(\xi) = \sum_{\alpha\in Z^d} M\Pi_n(\xi + 2\pi\alpha)M^{-1}f(2^{-n}(\xi + 2\pi\alpha))M\Pi_n^*(\xi + 2\pi\alpha)M^{-1} \tag{3.14}$$

and

$$\int_{T^d} (P_H^n f(\xi))g(\xi)d\xi = \int_{R^d} M\Pi_n(\xi)M^{-1}f(2^{-n}\xi)M\Pi_n^*(\xi)M^{-1}g(\xi)d\xi, \tag{3.15}$$

where

$$\Pi_n(\xi) = \prod_{j=1}^n H(2^{-j}\xi)\chi_{2^nT^d}(\xi), \quad n = 1, 2, \ldots. \tag{3.16}$$

Another crucial fact needed in what follows is the uniform integrability lemma by Long and Chen [26].

LEMMA 3.1. *Let* (X, μ) *be a* σ*-finite* (*nonnegative*) *measure space*, $\{f_n\}_n \subset L_+^1$, $f \in L_+^1$. *Assume that*

$$\underline{\lim}\ f_n \geqslant f, \quad \text{a.e.}, \lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu. \tag{3.18}$$

Then $\{f_n\}_n$ *converges to* f *in* L^1, *and, hence,* $\{f_n\}_n$ *is uniformly integrable.*

Proof. Since

$$\begin{aligned}\int_X f d\mu &= \int_X \min(\underline{\lim}\ f_n, f)d\mu = \int_X \underline{\lim}\min(f_n, f)d\mu\\ &\leqslant \underline{\lim}\int_X \min(f_n, f)d\mu \leqslant \overline{\lim}\int_X \min(f_n, f)d\mu \leqslant \int_X f d\mu,\end{aligned}$$

we derive

$$\int_X \min(f_n, f)d\mu \to \int_X f d\mu.$$

In the meanwhile,

$$f_n + f = \max(f_n, f) + \min(f_n, f)$$

implies that $\int_X \max(f_n, f)d\mu \to \int_X f d\mu$. Therefore,

$$\|f_n - f\|_1 = \int_X \max(f_n, f) - \min(f_n, f)d\mu \to 0.$$

The proof of the lemma is finished.

Now we are in the position to define $\hat{\varphi}$ and to characterize the orthonormality of $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k$.

Assume that $H(\xi) \in C(T^d, \mathcal{M}_m)$ and (2.7), (3.1), (3.8) hold. Then $\Pi_\infty(\xi) = \prod_1^\infty H(2^{-j}\xi)$ is also in $C(T^d, \mathcal{M}_m)$ and satisfies

$$M\Pi_\infty(\xi)M^{-1} = \begin{pmatrix} * & \\ \vdots & 0 \\ * & \end{pmatrix}. \tag{3.19}$$

Define

$$\hat{\varphi}(\xi) = \Pi_\infty(\xi)M^{-1}e_1. \tag{3.20}$$

It is easy to see that

$$M\Pi_\infty(\xi)\Pi_\infty^*(\xi)M^{-1} = M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}. \tag{3.21}$$

Our first fundamental result about the L^2-integrability of φ and the orthonormality of $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k$ is as follows.

THEOREM 3.1. *Assume that* $H(\xi) \in C(T^d, \mathcal{M}_m)$, *and* (2.7), (3.1), (3.8) *hold. Then* $\varphi \in L^2$. *Furthermore, assume that*

$$\Phi(\xi) = \sum_\alpha \hat{\varphi}(\xi + 2\pi\alpha)\hat{\varphi}^*(\xi + 2\pi\alpha) = ([\hat{\varphi}_i, \hat{\varphi}_j])_{i,j} \tag{3.22}$$

is continuous at $\xi = 0$. *Then* $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k$ *is orthonormal if and only if*

$$\Phi(\xi) \geqslant CI_m, \quad \text{a.e. } \xi. \tag{3.23}$$

Proof. For $\Pi_n(\xi)$ defined in (3.16), we have (by making use of (3.15))

$$\begin{aligned}\left(\frac{1}{2\pi}\right)^d \int_{R^d} M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}d\xi &= \left(\frac{1}{2\pi}\right)^d \int_{T^d} P_H^n I_m d\xi \\ &= \left(\frac{1}{2\pi}\right)^d \int_{T^d} I_m d\xi = I_m. \end{aligned}\tag{3.24}$$

Thus, for $i = 1, \ldots, m$,

$$\left(\frac{1}{2\pi}\right)^d \int_{R^d} e_i^* M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}e_i d\xi = 1.$$

Fatou's lemma gives

$$\left(\frac{1}{2\pi}\right)^d \int_{R^d} e_i^* M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}e_i d\xi \leqslant 1, \quad i = 1, \ldots, m.$$

That is to say

$$\int_{R^d} |\sum_j m_{i,j}\varphi_j(x)|^2 dx = \left(\frac{1}{2\pi}\right)^d \int_{R^d} |\sum_j m_{i,j}\hat{\varphi}_j(\xi)|^2 d\xi$$
$$\leqslant 1, \quad i = 1, \ldots, m. \tag{3.25}$$

This implies the L^2-integrability of φ, since M is unitary.

When $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k$ is orthonormal, then by Proposition 3.3, $\Phi(\xi) = I_m$ a.e. ξ; i.e., (3.23) holds. On the contrary, we have

$$\begin{aligned}\int_{R^d} M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)(\xi)M^{-1}d\xi &= \int_{T^d} M\Phi(\xi)M^{-1}d\xi \\ &= \int_{T^d} P_H^n(M\Phi(\xi)M^{-1})(\xi)d\xi \\ &= \int_{R^d} M\Pi_n(\xi)\Phi(2^{-n}\xi)\Pi_n^*(\xi)M^{-1}d\xi, \end{aligned} \tag{3.26}$$

and

$$\int_{R^d} \sum_{i=1}^m e_i^* M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)(\xi)M^{-1}e_i d\xi = \int_{R^d} \sum_{i=1}^m e_i^* M\Pi_n(\xi)\Phi(2^{-n}\xi)\Pi_n^*(\xi)M^{-1}e_i d\xi. \tag{3.27}$$

Notice that both of $M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)(\xi)M^{-1}$ and $M\Pi_n(\xi)\Phi(2^{-n}\xi)\Pi_n^*(\xi)M^{-1}$ are in M_m^+; hence,

$$f = \sum_{i=1}^m e_i^* M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)(\xi)M^{-1}e_i \in L_+^1$$

and

$$\{f_n\}_n = \left\{\sum_{i=1}^m e_i^* M\Pi_n(\xi)\Phi(2^{-n}\xi)\Pi_n^* M^{-1}e_i\right\}_n \subset L_+^1.$$

Notice that

$$(M\Phi(0)M^{-1})_{11} = 1,$$

owing to the fact

$$M\hat{\varphi}(0) = e_1, \quad M\hat{\varphi}(2\pi\alpha) = (0, *, \ldots, *)^t \quad \forall \alpha \in Z^d - \{0\}.$$

These can be seen from

$$M\hat{\varphi}(0) = M\Pi_\infty(0)M^{-1}e_1 = e_1$$

and for $\alpha \neq 0$ (writing $\alpha = 2^j\beta$, $\beta = 2\gamma + \nu$, $\nu \in E_d$)

$$e_1^* M\hat{\varphi}(2\pi\alpha) = e_1^* MH^j(0)M^{-1}MH(\nu\pi)M^{-1}M\hat{\varphi}(\beta\pi) = 0.$$

Since $\Phi(\xi)$ is continuous at $\xi = 0$, we have

$$\begin{aligned} \lim_{n\to\infty} f_n(\xi) &= \sum_{i=1}^m e_i^* M\Pi_\infty(\xi)\Phi(0)\Pi_\infty^*(\xi)M^{-1}e_i \\ &= \sum_{i=1}^m e_i^* M\Pi_\infty(\xi)\Pi_\infty^*(\xi)M^{-1}e_i \\ &= \sum_{i=1}^m e_i^* M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)(\xi)M^{-1}e_i = f(\xi). \end{aligned} \tag{3.28}$$

By making use of Lemma 3.1, (3.27), together with (3.28) implies the uniform integrability of $\{f_n\}_n$. Noticing that

$$\sum_{i=1}^m e_i^* M\Pi_n(\xi)\Phi(2^{-n}\xi)\Pi_n^*(\xi)M^{-1}e_i \geqslant C\sum_{i=1}^m e_i^* M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}e_i = Cg_n(\xi), \tag{3.29}$$

we see that $\{g_n\}_n$ is also uniformly integrable. This is a crucial fact, not only for this theorem, but also for the whole paper. On the other hand, we have

$$\begin{aligned} \left(\frac{1}{2\pi}\right)^d \int_{R^d} M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}e^{ik\cdot\xi}d\xi &= \left(\frac{1}{2\pi}\right)^d \int_{T^d} P_H^n(I_m)e^{ik\cdot\xi}d\xi \\ &= \delta_{0,k}I_m. \end{aligned} \tag{3.31}$$

Since $\{M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}\}_n$ is a uniformly integrable matrix family (it follows from the uniform integrability of $\{g_n\}$, and the fact that the (i, j)-entry of AA^* is dominated by $(AA^*)_{i,i}^{1/2}(AA^*)_{j,j}^{1/2}$), letting $n \to \infty$ in (3.31), we get

$$\left(\frac{1}{2\pi}\right)^d \int_{R^d} M\Pi_\infty(\xi)\Pi_\infty^*(\xi)M^{-1}e^{ik\cdot\xi}d\xi = \delta_{0,k}I_m,$$

which implies

$$\left(\frac{1}{2\pi}\right)^d \int_{R^d} M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}e^{ik\cdot\xi}d\xi = \delta_{0,k}I_m. \tag{3.32}$$

This is just what we want: $M\Phi(\xi)M^{-1} = I_m$; i.e., $\Phi(\xi) = I_m$. The proof is finished.

Remark. As in the classical case, Cohen's condition can be introduced. But, in the present case we can only show the sufficiency of Cohen's condition for (3.23). A set $K \subset R^d$ is said to be $2\pi Z^d$-congruent with T^d, if $|K| = (2\pi)^d$, and for any $\xi \in T^d$ there is $\alpha \in Z^d$, such that $\xi + 2\pi\alpha \in K$. $H(\xi) \in C(T^d, \mathsf{M}_m)$ is said to satisfy the Cohen's condition, if there is a compact K (not necessarily contains $\xi = 0$ as an inner point) which is $2\pi Z^d$-congruent with T^d such that

$$H(2^{-j}\xi)H^*(2^{-j}\xi) \geqslant C_0 I_m \quad \forall \xi \in K \quad \forall j = 1, 2, \ldots . \tag{3.33}$$

Now we want to show that the Cohen's condition implies (3.23) for all ξ, where $H(\xi) \in C(T^d, \mathsf{M}_m)$ satisfies (2.7), (3.1), (3.8), $\hat{\phi}(\xi) = \Pi_\infty(\xi)x$ (any $x \in R^d$), $\Phi(\xi) = ([\hat{\phi}_i, \hat{\phi}_j])_{i,j}$ is continuous at $\xi = 0$, and $\Phi(0) \geqslant C_1 I_m$. In fact, finding $n \in Z_+$ large enough, such that

$$|\Phi(2^{-n}\xi)| \geqslant \frac{C_1}{2} I_m, \quad \xi \in K.$$

Then we have

$$\begin{aligned} M\Phi(\xi)M^{-1} &= \sum_\nu MH\left(\frac{\xi}{2} + \nu\pi\right)\Phi\left(\frac{\xi}{2} + \nu\pi\right)H^*\left(\frac{\xi}{2} + \nu\pi\right)M^{-1} \\ &\geqslant MH\left(\frac{\xi}{2}\right)\Phi\left(\frac{\xi}{2}\right)H^*\left(\frac{\xi}{2}\right)M^{-1} \\ &\geqslant M\prod_1^n H(2^{-j}\xi)\Phi(2^{-n}\xi)\left(\prod_1^n H(2^{-j}\xi)\right)^*M^{-1} \\ &\geqslant \frac{C_1}{2} C_0^n I_m = CI_m \quad \forall \xi \in K. \end{aligned}$$

Hence for any $\eta \in T^d$, by writing $\eta = \xi + 2\pi\alpha$ ($\xi \in K$, $\alpha \in Z^m$),

$$\Phi(\eta) = \Phi(\xi + 2\pi\alpha) \geqslant CI_m.$$

This proves the assertion.

This result can be applied to some examples taken by Hérve [21]:

$$\text{Example 3, } H(\xi) = \begin{pmatrix} \cos^2\dfrac{\xi}{2} & \dfrac{1}{2}\sin\xi \\ \dfrac{1}{4}\sin\xi & \dfrac{1}{2} - \dfrac{1}{4}\cos\xi \end{pmatrix}, \ \det H(\xi) = \frac{1}{2}\cos^2\frac{\xi}{2},$$

$$\text{Example 4, } H(\xi) = \begin{pmatrix} \cos^2 \dfrac{\xi}{2} & -\dfrac{3}{4}\, i \sin \xi \\ \dfrac{i}{8} \sin \xi & \dfrac{1}{4} - \dfrac{1}{8} \cos \xi \end{pmatrix}, \ \det H(\xi) = \frac{1}{8} \cos^4 \frac{\xi}{2},$$

$$\text{Example 5, } H(\xi) = \begin{pmatrix} \cos^2 \dfrac{\xi}{2} & -\dfrac{15}{16}\, i \sin \xi & 0 \\ \dfrac{5i}{3^2} \sin \xi & \dfrac{1}{4} - \dfrac{7}{3^2} \cos \xi & -\dfrac{3}{8}\, i \sin \xi \\ \dfrac{1}{64} \cos \xi & -\dfrac{1}{64} \sin \xi & \dfrac{1}{8}\, i \sin \xi \end{pmatrix},$$

$$\det H(\xi) = 5 \cdot 2^{-9} \cos^6 \frac{\xi}{2}.$$

All of them satisfy the Cohen's condition with $K = [-\pi, \pi]$. And, hence, $C_2 I_m \geqslant \Phi(\xi) \geqslant C_1 I_m$ for all ξ.

As done by Long and Chen [26], we can give some other characterizations of (3.32) in terms of the eigenvalues and the eigenspaces of P_H. Notice that $H(\xi)$ is unchanged when $\hat{\varphi}(\xi)$ is replaced by $a\hat{\varphi}(\xi)$ for any $a \in C$.

THEOREM 3.2. *Let $H(\xi) \in C(T^d, \mathcal{M}_m)$ be such that* (2.7), (3.1), (3.8) *hold, and $\hat{\varphi}(\xi) = \Pi_\infty(\xi) M^{-1} e_1$, $\Phi(\xi) \in C(T^d, \mathcal{M}_m)$. Then* (3.32) *holds with $\hat{\varphi}$ replaced by $c\hat{\varphi}$ for some constant c, if and only if for the eigenvalue* 1, *the corresponding eigenspace K_1 of P_H* (*restricted on $C(T^d, \mathcal{M}_m)$*) *is of dimension* 1.

Proof. Since I_m and $M\Phi(\xi)M^{-1}$ are both the invariant matrices of P_H, so the condition K_1 is of dimension 1 implies that for positive c, we have $M\Phi(\xi)M^{-1} = c^{-2} I_m$, $\Phi(\xi) = c^{-2} I_m$; therefore $c\varphi$ makes (3.32) hold. Conversely, assume that $c\varphi$ makes (3.32) true; then $\{M\Pi_n(\xi)\Pi_n^*(\xi)\}_n$ is uniformly integrable. Suppose that K_1 is not of dimension 1, then P_H has another eigen matrix $G(\xi)$ and a constant ϵ such that

$$F(\xi) = \epsilon I_m + G(\xi) \in C_0(T^d, \mathcal{M}_m) \cap K_1, \quad F(\xi) \neq 0.$$

Thus we get (by making use of (3.15))

$$\begin{aligned} \int_{T^d} F(\xi) F^*(\xi) d\xi &= \int_{T^d} (P_H^n F(\xi)) F^*(\xi) d\xi \\ &= \int_{R^d} M\Pi_n(\xi) M^{-1} F(2^{-n}\xi) M \Pi_n^*(\xi) M^{-1} F^*(\xi) d\xi \\ &= \int_{R^d} A_n(\xi) d\xi. \end{aligned}$$

Notice that $\{A_n(\xi)\}$ is uniformly integrable (refer to the proof of Theorem 3.1) and

$$\lim_{n\to\infty} A_n(\xi) = M\Pi_\infty(\xi)M^{-1}F(0)M\Pi_\infty^*(\xi)M^{-1}F^*(\xi) = 0, \tag{3.34}$$

we have a contradiction. So K_1 must be of dimension 1. Equation (3.34) follows from an elementary calculation: $(ABC)_{i,j} = \Sigma_{k,l}\, a_{i,k}b_{k,l}c_{l,j} = 0$, where $a_{i,k} = 0$, $c_{l,j} = 0$, $k, l \geqslant 2$, $b_{1,1} = 0$. The proof is finished.

When $H(\xi)$ is a trigonometric polynomial, we can get a more precise characterization.

THEOREM 3.3. *Let $H(\xi)$ be a trigonometric polynomial such that* (2.7), (3.1), (3.8) *hold. Then* (3.32) *with $\hat{\varphi}$ replaced by $c\hat{\varphi}$ for some constant c is true, if and only if* 1 *is a uni-eigenvalue of P_H restricted on P.*

Proof. When 1 is a uni-eigenvalue, then the corresponding eigenspace K_1 is of dimension 1 (considering P_H as an operator restricted on P). Notice that $\varphi(x)$ is of compact support (see [21]), and in this case $\varphi(x) \in L^2(R^d)$, so $\Phi(\xi)$ is a trigonometric polynomial matrix, so $M\Phi(\xi)M^{-1} = cI_m$ and c can be 1 by normalizing $\hat{\varphi}$.

Assume that (3.32) is true. Then $\{M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}\}_n$ is uniformly integrable. Suppose that 1 is not a uni-eigenvalue. Then either K_1 was not of dimension 1, or there was a subspace of P and a basis $\{A_1, \ldots, A_m\}$, $m > 1$, of this subspace such that $A_1 = M\Phi M^{-1}$, and

$$P_HA_1 = A_1, \quad P_HA_2 = A_1 + A_2, \ldots .$$

The first case cannot occur as shown in Theorem 3.2. Suppose the second case occurs; then there were $\epsilon \neq 0$ such that

$$B = \epsilon A_1 + A_2 \in \mathsf{P}_0.$$

Then we would have $P_H^nB = (\epsilon + n)A_1 + A_2$, and hence,

$$\int_{T^d} P_H^nB(\xi)d\xi = \int_{T^d} ((\epsilon + n)A_1(\xi) + A_2(\xi))d\xi.$$

The left side tends to zero owing to the uniform integrability of $\{M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}\}_n$; meanwhile the right side tends to ∞. The contradiction completes the proof of the theorem.

4. ORTHOGONAL WAVELETS GENERATED BY ORTHOGONAL MRA(m)

Let $H(\xi) \in C(T^d, \mathsf{M}_m)$ be such that (2.7), (3.1), (3.8), (3.23) hold and $\Phi(\xi)$ is continuous at $\xi = 0$. Define $\hat{\varphi}(\xi)$ by (3.20). Notice that $\hat{\varphi}(0) \neq 0$, so $\{V_j\}_{-\infty}^{\infty}$ defined in (1.6) is a MRA(m) (see [2, 22, or 25]). And from Theorem 3.1, $\{V_j\}$ is an orthogonal MRA(m). Assume that there are $\{H_\mu(\xi)\}$, $\mu \in E_d - \{0\}$, in $C(T^d, \mathsf{M}_m)$, such that

$$\sum_\nu H_\mu(\xi + \nu\pi)H_{\mu'}^*(\xi + \nu\pi) = \delta_{\mu,\mu'}I_m \quad \forall \mu, \mu' \in E_d. \tag{4.1}$$

That is to say the $(m2^d \times m2^d)$-blocked matrix $(H_\mu(\xi + \nu\pi))_{\mu,\nu} \in E_d$ satisfies the equality

$$(H_\mu(\xi + \nu\pi))(H_\mu(\xi + \nu\pi)^* = \begin{pmatrix} I_m & & \\ & \ddots & \\ & & I_m \end{pmatrix}. \tag{4.2}$$

By expanding $(H_\mu(\xi + \nu\pi))$ and $(H_\mu(\xi + \nu\pi))^* = (H_\mu^*(\xi + \nu\pi))$, according to the usual rule, we see that $(H_\mu(\xi + \nu\pi))^*$ is the inverse of $(H_\mu(\xi + \nu\pi))$; i.e., we have

$$(H_\mu(\xi + \nu\pi))^*(H_\mu(\xi + \nu\pi)) = \begin{pmatrix} I_m & & \\ & \ddots & \\ & & I_m \end{pmatrix}. \tag{4.3}$$

That is to say, the duality of (4.1) holds, too:

$$\sum_\mu H_\mu^*(\xi + \nu\pi)H_\mu(\xi + \nu'\pi) = \delta_{\nu,\nu'}I_m \quad \forall \nu, \nu' \in E_d. \tag{4.4}$$

Now define

$$\hat{\psi}_\mu(\xi) = H_\mu\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right), \quad \mu \in E_d - \{0\}. \tag{4.5}$$

Notice that $\hat{\varphi}(\xi)$ can be written as $\hat{\psi}_0(\xi)$. We want to show that $\{\psi_\mu(x)\} = \{\psi_{1,\mu}(x), \ldots, \psi_{m,\mu}(x)\}$, $\mu \in E_d - \{0\}$, is the orthonormal wavelets we wanted.

LEMMA 4.1. *Let $\hat{\varphi}(\xi)$ be the scaling function vector of an orthogonal* MRA(m), $\{H_\mu(\xi)\} \subset C(T^d, M_m)$, $\mu \in E_d - \{0\}$, $k \in Z^d$, $\{\hat{\Psi}_\mu\}$ *is defined as in* (4.5). *Then* $\{\Psi_{r,\mu}(x - k)_{r,\mu,k}$ *is orthonormal if and only if* $H(\xi) = (H_\mu(\xi + \nu\pi))_{\mu,\nu}$ *satisfies* (4.1).

Proof. The proof is almost the same as in the classical case. We have

$$\begin{aligned}\int_{R^d} \psi_\mu(x)\psi_{\mu'}^*(x - k)dx &= \left(\frac{1}{2\pi}\right)^d \int_{R^d} \hat{\psi}_\mu(\xi)\hat{\psi}_{\mu'}^*(\xi)e^{ik\cdot\xi}d\xi \\ &= \left(\frac{1}{2\pi}\right)^d \int_{T^d} \sum_\nu H_\mu\left(\frac{\xi}{2} + \nu\pi\right)H_{\mu'}^*\left(\frac{\xi}{2} + \nu\pi\right)e^{ik\cdot\xi}d\xi.\end{aligned}$$

Thus the left side equals $\delta_{\mu,\mu'}\delta_{0,k}I_m$ if, and only if, $\sum_\nu H_\mu(\xi/2 + \nu\pi)H_\mu'^*(\xi/2 + \nu\pi) = \delta_{\mu,\mu'}I_m$. The proof is finished.

In order to show that such $\{\psi_{1,\mu}(x - k), \ldots, \psi_{m,\mu}(x - k)\}$, $\mu \in E_d - \{0\}$, $k \in Z^d$, spans $W_0 = V_1 \ominus V_0$, indeed, we introduce the projection operators P_j, Q_j. For $f \in L^2(R^d)$, define

$$P_j f(x) = \sum_{r,k} \langle f, \varphi_{r,j,k} \rangle \varphi_{r,j,k}(x), \quad j \in Z, \tag{4.6}$$

and

$$Q_j f(x) = \sum_{\mu \neq 0} \sum_{r,k} \langle f, \psi_{r,\mu,j,k} \rangle \psi_{r,\mu,j,k}(x), \quad j \in Z, \tag{4.7}$$

where j, k are the dilation and translation indices, respectively.

LEMMA 4.2. *We have*

$$P_1 f = P_0 f + Q_0 f \quad \forall f \in L^2(R^d). \tag{4.8}$$

Proof. It is enough to prove

$$\langle P_1 f, g \rangle = \langle P_0 f, g \rangle + \langle Q_0 f, g \rangle \quad \forall f, g \in L^2(R^d); \tag{4.9}$$

i.e.,

$$\begin{aligned} I &= \sum_{r,k} \langle f, 2^{d/2} \varphi_r(2\ \cdot - k) \rangle \langle g, 2^{d/2} \varphi_r(2\ \cdot - k) \rangle^- \\ &= \sum_{r,\mu,k} \langle f, \psi_{r,\mu}(\cdot - k) \rangle \langle g, \psi_{r,\mu}(\cdot - k) \rangle^- = II. \end{aligned} \tag{4.10}$$

We have

$$\begin{aligned} II &= \left(\frac{1}{2\pi}\right)^{2d} \sum_r \sum_\mu \sum_k \int_{R^d} \hat{f}(\xi) \overline{\hat{\psi}_{r,\mu}}(\xi) e^{ik\cdot\xi} d\xi (\cdots)^- \\ &= \left(\frac{1}{2\pi}\right)^{2d} \sum_r \sum_\mu \sum_k \int_{T^d} \sum_\alpha \hat{f}(\xi + 2\pi\alpha) \overline{\hat{\psi}_{r,\mu}}(\xi + 2\pi\alpha) e^{ik\xi} d\xi (\cdots)^- \\ &= \left(\frac{1}{2\pi}\right)^{d} \sum_r \sum_\mu \int_{T^d} \sum_\alpha \hat{f}(\xi + 2\pi\alpha) \overline{\hat{\psi}_{r,\mu}}(\xi + 2\pi\alpha) \\ &\quad \times \sum_\beta \hat{g}(\xi + 2\pi\beta) \hat{\psi}_{r,\mu}(\xi + 2\pi\beta) d\xi \\ &= \left(\frac{1}{2\pi}\right)^{d} \sum_\mu \int_{T^d} \sum_{\alpha,\beta} \hat{f}(\xi + 2\pi\alpha) \hat{\psi}_\mu^*(\xi + 2\pi\alpha) \hat{\psi}_\mu(\xi + 2\pi\beta) \hat{g}(\xi + 2\pi\beta) d\xi \\ &= \left(\frac{1}{2\pi}\right)^{d} \int_{T^d} \sum_{\nu,\nu'} \sum_{\alpha',\beta'} \hat{f}(\xi + 2\pi\nu + 4\pi\alpha') \hat{\varphi}_\mu^* \left(\frac{\xi}{2} + \nu\pi + 2\pi\alpha'\right) \\ &\quad \times \sum_\mu m_\mu^* \left(\frac{\xi}{2} + \nu\pi\right) m_\mu \left(\frac{\xi}{2} + \nu'\pi\right) \hat{\varphi}\left(\frac{\xi}{2} + \nu'\pi + 2\pi\beta'\right) \hat{g}(\xi + 2\pi\nu' + 4\pi\beta') d\xi \end{aligned}$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{T^d} \sum_{\nu} \sum_{\alpha',\beta'} \hat{f}(\xi + 2\pi\nu + 4\pi\alpha')\hat{\varphi}^*\left(\frac{\xi}{2} + \nu\pi + 2\pi\alpha'\right)$$

$$\times \hat{\varphi}\left(\frac{\xi}{2} + \nu\pi + 2\pi\beta'\right)\hat{g}(\psi + 2\pi\nu' + 4\pi\beta')d\xi$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{[0,2\pi]^d} = \left(\frac{1}{2\pi}\right)^d \sum_{\nu} \int_{[0,2\pi]^d+2\pi\nu}$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{2T^d} \sum_{\alpha,\beta} \hat{f}(\xi + 4\pi\alpha)\hat{\varphi}^*\left(\frac{\xi}{2} + 2\pi\alpha\right)\hat{\varphi}\left(\frac{\xi}{2} + 2\pi\beta\right)\hat{g}^*(\xi + 4\pi\beta)d\xi,$$

and

$$I = \left(\frac{1}{2\pi}\right)^{2d} \frac{1}{2^d} \sum_{r,k} \int_{R^d} \hat{f}(\xi)\overline{\hat{\varphi}_r}\left(\frac{\xi}{2}\right)e^{ik\cdot\xi/2}d\xi(\cdots)^-$$

$$= \left(\frac{1}{2\pi}\right)^d\left(\frac{1}{4\pi}\right)^d \sum_{r,k} \int_{2T^d} \sum_{\alpha} \hat{f}(\xi + 4\pi\alpha)\overline{\hat{\varphi}_r}\left(\frac{\xi}{2} + 2\pi\alpha\right)e^{ik\cdot\xi/2}d\xi(\cdots)^-$$

$$= \left(\frac{1}{2\pi}\right)^d \sum_{r} \int_{2T^d} \sum_{\alpha,\beta} \hat{f}(\xi + 4\pi\alpha)\overline{\hat{\varphi}_r}\left(\frac{\xi}{2} + 2\pi\alpha\right)\hat{\varphi}_r\left(\frac{\xi}{2} + 2\pi\beta\right)\hat{g}(\xi + 4\pi\beta)d\xi$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{2T^d} \sum_{\alpha,\beta} \hat{f}(\xi + 4\pi\alpha)\hat{\varphi}^*\left(\frac{\xi}{2} + 2\pi\alpha\right)\hat{\varphi}\left(\frac{\xi}{2} + 2\pi\beta\right)\hat{g}(\xi + 4\pi\beta)d\xi.$$

This completes the proof of the lemma.

Combining the two lemmas, we get

THEOREM 4.1. *Let* $H_0(\xi) \in C(\Pi^d, \mathcal{M}_m)$ *be such that* (2.7), (3.1), (3.8), (3.23) *hold. Assume* $\{H_\mu(\xi)\}$, $\mu \in E_d - \{0\}$, *satisfies* (4.1). *Then* $\{\psi_{r,\mu,j,k}\}$ *constructed as in* (4.5) *is an orthonormal wavelet basis of* $L^2(R^d)$.

Proof. From Lemma 4.1, we know that $\{\psi_{r,\mu,j,k}\}$ is orthonormal. Furthermore, for $f \in L^2(R^d)$, we have

$$\lim_{j\to-\infty} P_j f = 0, \quad \lim_{j\to\infty} P_j f = f.$$

Hence

$$f = \lim_{j\to\infty} (P_j f - P_{-j} f) = \sum_{\mu\neq 0} \sum_{r,j,k} \langle f, \psi_{r,\mu,j,k}\rangle\psi_{r,\mu,j,k}(x).$$

The proof of the theorem is finished.

5. BIORTHOGONAL VERSIONS

Assume that we are given a couple $\{H_0(\xi); \tilde{H}_0(\xi)\}$ in $C(T^d, \mathsf{M}_m)$ satisfying

$$\sum_{\nu} H_0(\xi + \nu\pi)\tilde{H}_0^*(\xi + \nu\pi) = I_m \tag{5.1}$$

with

$$H_0(0) = M^{-1}\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} M, \quad \tilde{H}_0(0) = M^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \tilde{D} \end{pmatrix} M, \tag{5.2}$$

where $M = (m_{i,j})$ is a unitary matrix, and

$$D = \begin{pmatrix} \lambda_2 & \mu_2 & & \\ & \ddots & \ddots & \\ & & \ddots & \mu_{m-1} \\ & & & \lambda_m \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} \tilde{\lambda}_2 & \tilde{\mu}_2 & & \\ & \ddots & \ddots & \\ & & \ddots & \tilde{\mu}_{m-1} \\ & & & \tilde{\lambda}_m \end{pmatrix}, \tag{5.3}$$

$$\|H_0(\xi) - H_0(0)\| + \|\tilde{H}_0(\xi) - \tilde{H}_0(0)\| \leqslant C|\xi|^{\epsilon}, \quad \epsilon > 0, \tag{5.4}$$

$$MH_0^*(\nu\pi)M^{-1}e_1 = \delta_{0,\nu}e_1, \quad M\tilde{H}_0^*(\nu\pi)M^{-1}e_1 = \delta_{0,\nu}e_1. \tag{5.5}$$

Notice that the condition (5.5) holds automatically when

$$\Phi(\nu\pi) \geqslant cI_m, \quad \tilde{\Phi}(\nu\pi) \geqslant cI_m \tag{5.6}$$

is assumed. This can be seen from $MH_0^*(0)M^{-1}e_1 = \mathrm{e}_1$ and

$$\begin{aligned} e_1^*M\Phi(0)M^{-1}e_1 &= e_1^* \sum_{\nu} MH_0(\nu\pi)M^{-1}M\Phi(\nu\pi)M^{-1}MH_0^*(\nu\pi)M^{-1}e_1 \\ &\geqslant e_1^*M\Phi(0)M^{-1}e_1 + c \sum_{\nu\neq 0} |H_0^*(\nu\pi)M^{-1}e_1|. \end{aligned}$$

The introduction of (5.5) is to guarantee the invariance of C_0, P_0 under the action of P_{H_0} and $P_{\tilde{H}_0}$. Assume as well that $\Phi(\xi)$, $\tilde{\Phi}(\xi)$ are continuous and

$$\Phi(\xi) \geqslant CI_m, \quad \tilde{\Phi}(\xi) \geqslant CI_m, \quad \text{for all } \xi. \tag{5.7}$$

Furthermore, assume that we have matrix-extensions

$$H(\xi) = (H_\mu(\xi + \nu\pi))_{\mu,\nu}, \quad \tilde{H}(\xi) = (\tilde{H}_\mu(\xi + \nu\pi))_{\mu,\nu}, \tag{5.9}$$

which satisfy

$$(H_\mu(\xi+\nu\pi))_{\mu,\nu}(\tilde H_\mu(\xi+\nu\pi))^*_{\mu,\nu} = \begin{pmatrix} I_m & & \\ & \ddots & \\ & & I_m \end{pmatrix}$$
$$= (\tilde H_\mu(\xi+\nu\pi))^*_{\mu,\nu}(H_\mu(\xi+\nu\pi))_{\mu,\nu};$$

more precisely,

$$\sum_\nu H_\mu(\xi+\nu\pi)\tilde H^*_{\mu'}(\xi+\nu\pi) = \delta_{\mu,\mu'}I_m, \quad \mu,\mu'\in E_d, \tag{5.10}$$

and

$$\sum_\mu \tilde H^*_\mu(\xi+\nu\pi)H_\mu(\xi+\nu'\pi) = \delta_{\nu,\nu'}I_m, \quad \nu,\nu'\in E_d. \tag{5.11}$$

Finally, assume that

$$|H_\mu(\xi)\hat\varphi(\xi)| + |\tilde H_\mu(\xi)\hat{\tilde\varphi}(\xi)| \leqslant C|\xi|^\epsilon \quad \text{for } |\xi|\leqslant 1. \tag{5.12}$$

Define

$$\hat\varphi(\xi) = \prod^\infty H_0(2^{-j}\xi)M^{-1}e_1, \quad \hat{\tilde\varphi}(\xi) = \prod^\infty \tilde H_0(2^{-j}\xi)M^{-1}e_1, \tag{5.13}$$

$$\hat\psi_\mu(\xi) = H_\mu\left(\frac{\xi}{2}\right)\hat\varphi\left(\frac{\xi}{2}\right), \quad \hat{\tilde\psi}_\mu(\xi) = \tilde H_\mu\left(\frac{\xi}{2}\right)\hat{\tilde\varphi}\left(\frac{\xi}{2}\right), \quad \mu\in E_d-\{0\}. \tag{5.14}$$

Under the preceding conditions, we have $\hat\varphi\in L^2$, $\hat{\tilde\varphi}\in L^2$ no longer (as shown in the classical case). But if we take $\hat\varphi\in L^2$, $\hat{\tilde\varphi}\in L^2$ for granted, then we can get the biorthogonal versions of Theorems 3.1, 3.2, 3.3 routinely. The biorthogonal version of Theorem 4.1 should be read as $\{\psi_{r,\mu,j,k};\tilde\psi_{r,\mu,j,k}\}$, constructed above as a biorthogonal system of $L^2(R^d)$, and we have (in L^2-convergence sense)

$$f = \sum\langle f,\psi_{r,\mu,j,k}\rangle\tilde\psi_{r,\mu,j,k} = \sum\langle f,\tilde\psi_{r,\mu,j,k}\rangle\psi_{r,\mu,j,k}. \tag{5.15}$$

But $\{\psi_{r,\mu,j,k}\}$ or $\{\tilde\psi_{r,\mu,j,k}\}$ can fail to be a Riesz basis of $L^2(R^d)$. The main task of this section is to generalize the results of Cohen and Daubechies [5] (it has been generalized to the case $d\geqslant 1$, $m=1$, by Long and Chen [26]) to MRA(m) ($d\geqslant 1$, $m\geqslant 1$), which can give the L^2-integrability of $\hat\varphi(\hat{\varphi})$, the Riesz basis property of $\{\psi_{r,\mu,j,k}\}(\{\tilde\psi_{r,\mu,j,k}\})$, and the uniform integrability of $\{\Pi_n(\xi)\Pi_n^*(\xi)\}_n$ $(\{\tilde\Pi_n(\xi)\tilde\Pi_n^*(\xi)\}_n)$, by making use of the eigenvalue estimates of P_{H_0} $(P_{\tilde H_0})$ restricted on $\mathcal{P}_0$ when $H_0(\xi)$ $(\tilde H_0(\xi))$ is a trigonometric polynomial.

At first, we want to show that the eigenvalue estimates we will use in what follows is necessary in some sense.

THEOREM 5.1. *Let* $H_0(\xi)(\tilde{H}_0(\xi)) \in C(T^d, \mathcal{M}_m)$ *be such that* (5.2), (5.3), (5.4), (5.7) *hold, and* $\Phi(\tilde{\Phi}) \in L^1$. *Then any eigenvalue* λ *of* $P_{H_0}(\tilde{\lambda}$ *of* $P_{\tilde{H}_0})$, *restricted on* $C(T^d, \mathcal{M}_m)$, *satisfies* $|\lambda| \leqslant 1$, *and any eigenvalue* λ *of* $P_{H_0}(\tilde{\lambda}$ *of* $P_{\tilde{H}_0})$, *restricted on* $C_0(T^d, \mathcal{M}_m)$, *satisfies* $|\lambda| < 1$.

Proof. Only prove the assertion for P_{H_0}. Assume that λ is an eigenvalue of P_{H_0} restricted on $C(T^d, \mathcal{M}_m)$, and $f(\xi) \in C(T^d, \mathcal{M}_m)(\neq 0)$ satisfies $P_{H_0} f = \lambda f$. Then for $n \in Z_+$, we have

$$\begin{aligned}\lambda^n \int_{T^d} f(\xi) f^*(\xi) d\xi &= \int_{T^d} (P^n_{H_0} f(\xi)) f^*(\xi) d\xi \\ &= \int_{R^d} M\Pi_n(\xi) M^{-1} f(2^{-n}\xi) M\Pi_n^*(\xi) M^{-1} f^*(\xi) d\xi \\ &= \int_{R^d} G_n(\xi) d\xi. \end{aligned} \tag{5.16}$$

Notice that, under the conditions (5.2), (5.3), (5.4), (5.7), $\{\int_{R^d} G_n(\xi) d\xi\}_n$ is bounded. So $|\lambda| \leqslant 1$, since $\int_{T^d} f(\xi) f^*(\xi) d\xi \neq 0$. Consider P_{H_0} as an operator acting on $C_0(T^d, \mathcal{M}_m)$. The preceding argument shows that $|\lambda| < 1$, since $\{G_n(\xi)\}_n$ is uniformly integrable and $\lim_{n\to\infty} G_n(\xi) = 0$, a.e. ξ. The proof of the theorem is finished.

Remark. In the case $d = m = 1$, the result is due to Cohen and Daubechies [5]. But the proof is a little complicated and is not available in the case $d > 1$. Here the proof is given by Long and Chen [26].

A natural question is that from $\{|\lambda| < 1$: λ's all eigenvalues of P_{H_0} restricted on $\mathcal{P}_0\}$, what can we get, when $H_0(\xi)$ is a trigonometric polynomial? As shown in [5], from $|\lambda| < 1$ we can get enough decay of $\hat{\varphi}$ and, hence, the L^2-integrability of $\hat{\varphi}$, the L^2-convergence of $\{\hat{\varphi}_n\}_n$ (the substitution of $\{\Pi_n(\xi)\}_n$ when $m = 1$), and the Riesz basis property of $\{\psi_{\mu,j,k}\}$ with the help of (5.12). Now we want to establish the same result in the case of MRA(m), $d \geqslant 1$, $m \geqslant 1$. The main idea is similar, but some modifications should be done (some of them has been done in [25]).

In the case $H(\xi)$ is a trigonometric polynomial, $\mathcal{P}_0$ is a linear space of finite dimension. Define the norm in $\mathcal{P}_0$ by

$$\|A(\cdot)\|_* = \sum_{i,j} \|a_{i,j}(\cdot)\|_\infty \quad \text{for } A(\xi) = (a_{i,j}(\xi)) \in \mathcal{P}_0. \tag{5.17}$$

Denote the spectral radius of P_H restricted on $\mathcal{P}_0$ by $\rho(P_H)$. We have

$$\lim \|P_H^n\|^{1/n}_{(\mathcal{P}_0, \mathcal{P}_0)} = \rho(P_H) = \max\{|\lambda|: \lambda\text{'s all eigenvalues of } P_H\}. \tag{5.18}$$

So, when all eigenvalues λ's of P_H satisfy $|\lambda| < 1$, then there is a $\rho < 1$ such that for n large enough, we have

$$\|P_H^n A\|_* \leqslant \|P_H^n\|_{(\mathcal{P}_0,\mathcal{P}_0)}\|A\|_* \leqslant \rho^n\|A\|_* \quad \forall A \in \mathcal{P}_0. \tag{5.19}$$

Now we are in the position to give another main result in the section as follows.

THEOREM 5.2. *Let $H(\xi)$ be a trigonometric polynomial such that* (5.2), (5.3), (5.4), *and* (5.5) *hold. Assume that all eigenvalues λ's of P_H restricted on $\mathcal{P}_0$ are in the unit circle. Then $\hat{\varphi} \in L^2$ and $\{\Pi_n(\xi)\Pi_n^*(\xi)\}_n$ is uniformly integrable, and there is a $\delta > 0$ such that*

$$\sum_{\alpha} |\hat{\varphi}(\xi + 2\pi\alpha)|^{2-\delta} \leqslant C, \quad \text{a.e. } \xi, \tag{5.20}$$

$$|\hat{\varphi}(\xi)| \leqslant C(1 + |\xi|)^{-\delta} \quad \forall\xi. \tag{5.21}$$

Proof. For the sake of completeness, we give the proof in detail. Set

$$\theta(\xi) = ((1 - \cos\xi_1)^2 + \cdots + (1 - \cos\xi_d)^2)I_m, \quad \xi = (\xi_1, \ldots, \xi_d). \tag{5.22}$$

Then $\theta(\xi) \in \mathcal{P}_0$ (since $\theta(0) = 0$), and

$$\theta(\xi) \geqslant CI_d \quad \text{for } \xi \in \left\{\frac{\pi}{2} \leqslant |\xi| \leqslant \pi\right\}.$$

For n large enough, we have

$$\begin{aligned}
&\left\|\int_{R^d} M\Pi_n(\xi)M^{-1}\theta(2^{-n}\xi)M\Pi_n^*(\xi)M^{-1}d\xi\right\|_* \\
&\qquad = \left\|\int_{T^d} P_H^n\theta(\xi)d\xi\right\|_* \leqslant O(1)\|P_H^n\theta\|_* \leqslant O(1)\rho^n\|\theta\|_* \leqslant O(1)\rho^n.
\end{aligned} \tag{5.23}$$

Since on $\Delta_n = \{\xi: 2^{n-1}\pi \leqslant |\xi| \leqslant 2^n\pi\}$, we have $\theta(2^{-n}\xi) \geqslant cI_m$. Therefore, we get

$$\begin{aligned}
&\left\|\int_{\Delta_n} M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}d\xi\right\|_* \\
&\quad \leqslant O(1)\sum_i e_i^* \int_{\Delta_n} M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}d\xi e_i \\
&\quad \leqslant O(1)\sum_i e_i^* \int_{\Delta_n} M\Pi_n(\xi)M^{-1}\theta(2^{-n}\xi)M\Pi_n^*(\xi)M^{-1}d\xi e_i \\
&\quad \leqslant O(1)\sum_i \int_{R^d} e_i^* M\Pi_n(\xi)M^{-1}\theta(2^{-n}\xi)M\Pi_n^*(\xi)M^{-1}d\xi e_i \\
&\quad \leqslant O(1)\left\|\int_{R^d} M\Pi_n(\xi)M^{-1}\theta(2^{-n}\xi)M\Pi_n^*(\xi)M^{-1}d\xi\right\|_* \leqslant O(1)\rho^n.
\end{aligned} \tag{5.24}$$

Noticing as well that on 2^nT^d, we have

$$\hat{\varphi}(\xi) = \Pi_n(\xi)\hat{\varphi}(2^{-n}\xi), \quad \max_{|\xi|\leqslant\pi/2} |\hat{\varphi}(\xi)| \leqslant O(1).$$

Hence, we get

$$\begin{aligned}\int_{\Delta_n} e_i^* M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}e_i d\xi &= \int_{\Delta_n} e_i^* M\Pi_n(\xi)\hat{\varphi}(2^{-n}\xi)\hat{\varphi}^*(2^{-n}\xi)\Pi_n^*(\xi)M^{-1}e_i d\xi \\ &\leqslant O(1)\int_{\Delta_n} e_i^* M\Pi_n(\xi)\Pi_n^*(\xi)M^{-1}e_i d\xi \leqslant O(1)\rho^n \quad (5.25)\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\Delta_n} |\hat{\varphi}(\xi)|^2 d\xi &= \int_{\Delta_n} \sum_i e_i^*\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)e_i d\xi \\ &\leqslant O(1)\int_{\Delta_n} \sum_i e_i^* M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}e_i d\xi \\ &\leqslant O(1)\rho^n, \end{aligned} \tag{5.26}$$

which gives the L^2-integrability of $\hat{\varphi}$ immediately.

Now we deduce (5.20), (5.21) from (5.26). Up to now, we have seen that $\hat{\varphi} \in L^2$, and $\varphi(x)$ has a compact support Ω. Select a Schwartz function $\epsilon(x)$ satisfies $\epsilon(x)|_\Omega = 1$. Thus, we have

$$\hat{\varphi}(\xi) = \left(\frac{1}{2\pi}\right)^d \int_{R^d} \hat{\varphi}(\eta)\hat{\epsilon}(\xi - \eta)d\eta,$$

and, hence (for some m large enough),

$$\begin{aligned}|\hat{\varphi}(\xi)| &\leqslant \left(\int_{\{|\eta|\leqslant|\xi|/2\}} + \int_{\{|\eta|>|\xi|/2\}}\right)|\hat{\varphi}(\eta)\|\hat{\epsilon}(\xi - \eta)|d\eta \\ &\leqslant C_m|\eta|^{-m}|\xi|^{d/2}\|\hat{\varphi}\|_2 + O(1)\left(\int_{\{|\eta|>|\xi|/2\}}|\hat{\varphi}(\eta)|^2 d\eta\right)^{1/2} \\ &\leqslant C|\xi|^{-\delta}.\end{aligned}$$

Inequality (5.21) has thus been proved. For the proof of (5.20), we need the so-called Plancherel, Polya, and Nikolski's inequality: Let $f \in L^p$ be such that supp $\hat{f} \subset \Omega$ (some fixed compact set). Let $h > 0$, $k = (k_1, \ldots, k_d) \in Z^d$, $Q_k^h = \{x\colon hk_j \leqslant x_j < h(k_j + 1)\}$, $x_k \in Q_k^h$, $1 \leqslant p \leqslant \infty$. Then

$$\sum_k |f(x_k)|^p \leqslant C\int_{R^d} |f(x)|^p dx, \tag{5.27}$$

with C independent of f. Applying (5.27) to $f(\xi) = \hat{\varphi}(\xi)$ (a vector function) and $p = 2 - \delta$ ($\delta > 0$ determined later), and setting $\Delta_j = \{2^j\pi \leqslant |\xi| < 2^{j+1}\pi\}$, we get

$$\begin{aligned}\sum_{\alpha} |\hat{\varphi}(\xi + 2\pi\alpha)|^{2-\delta} &\leqslant C \int_{R^d} |\hat{\varphi}(\xi)|^{2-\delta} d\xi = C + C \sum_{j=1}^{\infty} \int_{\Delta_j} \\ &\leqslant C + C \sum_{j=1}^{\infty} \left(\int_{\Delta_j} |\hat{\varphi}(\xi)|^2 d\xi \right)^{(2-\delta)/2} 2^{(d\delta/2)j} \\ &= C + C \sum_{1}^{\infty} \rho^{(2-\delta)/2j} 2^{(d\delta/2)j} \leqslant C,\end{aligned}$$

provided $\delta > 0$ being small enough such that $\rho^{(2-\delta)/2} 2^{d\delta/2} < 1$.

The uniform integrability of $\{\Pi_n(\xi)\Pi_n^*(\xi)\}_n$ under the conditions (5.2), (5.3), (5.4), and (5.5) has been shown by Long and Mo [28]. It should be appreciated for the kindness of Long and Mo [28] to permit us to sketch the proof here. Denote

$$\hat{\eta}_n(\xi) = M\Pi_n(\xi)M^{-1}, \quad \theta^{(1)}(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \theta^{(2)}(\xi) = \begin{pmatrix} 0 & 0 \\ 0 & I_{r-1} \end{pmatrix},$$

$$\hat{\eta}_n^{(1)}(\xi) = \hat{\eta}_n(\xi)\theta^{(1)}(\xi), \quad \hat{\eta}_n^{(2)}(\xi) = \hat{\eta}_n(\xi)\theta^{(2)}(\xi).$$

Then,

$$\hat{\eta}_n^{(1)}(\xi)\hat{\eta}_n^{(2)*}(\xi) = 0 = \hat{\eta}_n^{(2)}(\xi)\hat{\eta}_n^{(1)*}(\xi), \quad \hat{\eta}_n(\xi)\hat{\eta}_n^*(\xi) = \hat{\eta}_n^{(1)}(\xi)\hat{\eta}_n^{(1)*}(\xi) + \hat{\eta}_n^{(2)}(\xi)\hat{\eta}_n^{(2)*}(\xi).$$

So it is enough to prove the uniform integrability of $\{\hat{\eta}_n^{(i)}(\xi)\hat{\eta}_n^{(i)*}(\xi)\}_n$, $i = 1, 2$. From (5.23), we have

$$\begin{aligned}\left\| \int_{R^d} \hat{\eta}_n^{(2)}(\xi)\hat{\eta}_n^{(2)*}(\xi)d\xi \right\|_* &= \left\| \int_{R^d} \hat{\eta}_n(\xi)\theta^{(2)}(2^{-n}\xi)\theta^{(2)*}(2^{-n}\xi)\hat{\eta}_n^*(\xi)d\xi \right\|_* \\ &= \left\| \int_{R^d} \hat{\eta}_n(\xi)\theta^{(2)}(2^{-n}\xi)\hat{\eta}_n^*(\xi)d\xi \right\| \leqslant O(1)\rho^n, \end{aligned} \tag{5.28}$$

which implies that $\{\hat{\eta}_n^{(2)}(\xi)\hat{\eta}_n^{(2)*}(\xi)\}_n$ converges to zero in L^1. Meanwhile, we have

$$\int_{R^d} M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}d\xi = \int_{R^d} \hat{\eta}_n(\xi)M\Phi(2^{-n}\xi)M^{-1}\hat{\eta}_n^*(\xi)d\xi = \int_{R^d} G_n(\xi)d\xi,$$

and (by using $(M\Phi(0)M^{-1})_{11} = 1$)

$$\begin{aligned}\lim_{n\to\infty} G_n(\xi) &= M\Pi_\infty(\xi)M^{-1}(M\Phi(0)M^{-1})_{11}M\Pi_\infty^*(\xi)M^{-1} \\ &= M\Pi_\infty(\xi)\Pi_\infty^*(\xi)M^{-1} = M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1},\end{aligned}$$

which implies that $\{G_n(\xi)\}_n$ converges to $M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}$ in L^1. Denote

$$\theta^{(3)}(\xi) = M\Phi(\xi)M^{-1} - \theta^{(1)}(\xi).$$

Then $\theta^{(3)}(\xi) \in \mathsf{P}_0$, and, hence,

$$\left\| \int_{R^d} \hat{\eta}_n(\xi)\theta^{(3)}(2^{-n}\xi)\hat{\eta}_n^*(\xi)d\xi \right\| \leqslant c\rho^n.$$

Thus we get

$$\begin{aligned} \int_{R^d} G_n(\xi)d\xi - \int_{R^d} \hat{\eta}_n(\xi)\theta^{(1)}(2^{-n}\xi)\hat{\eta}_n^*(\xi)d\xi \to 0, \\ \lim_{n\to\infty} \int_{R^d} \hat{\eta}_n(\xi)\theta^{(1)}(2^{-n}\xi)\theta^{(1)*}(2^{-n}\xi)\hat{\eta}_n^*(\xi)d\xi &= \lim_{n\to\infty} \int_{R^d} G_n(\xi)d\xi \\ &= \int_{R^d} M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}d\xi, \end{aligned}$$

and

$$\lim_{n\to\infty} \hat{\eta}_n(\xi)\theta^{(1)}(2^{-n}\xi)\theta^{(1)*}(2^{-n}\xi)\hat{\eta}_n^*(\xi) = M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}, \tag{5.29}$$

which implies that $\{\hat{\eta}_n^{(1)}(\xi)\hat{\eta}_n^{(1)*}(\xi)\}_n$ converges to $M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}$ in L^1. In a word, $\{\hat{\eta}_n(\xi)\hat{\eta}_n^*(\xi)\}_n$ converges to $M\hat{\varphi}(\xi)\hat{\varphi}^*(\xi)M^{-1}$ in L^1. The proof of the theorem is finally complete.

Now we deduce a theorem to get biorthogonal MRA(m) from Theorem 5.2.

THEOREM 5.3. *Let $H_0(\xi)$, $\tilde{H}_0(\xi)$ be two trigonometric polynomials satisfying the conditions* (5.1), (5.2), (5.3), (5.4), *and* (5.5). *Furthermore, assume that the eigenvalues, λ's of P_{H_0} and $\tilde{\lambda}$'s of $P_{\tilde{H}_0}$, restricted on P_0, are all in the unit circle. Then $\{\varphi(x); \tilde{\varphi}(x)\}$ generates a biorthogonal* MRA(m).

Proof. From the conditions imposed on $H_0(\xi)$, $\Phi(\xi)$, and

$$\int_{T^d} M\Phi(\xi)M^{-1}d\xi = \int_{R^d} M\Pi_n(\xi)\Phi(2^{-n}\xi)\Pi_n^*(\xi)M^{-1}d\xi = \int_{R^d} G_n(\xi)d\xi,$$

we see that $\{G_n(\xi)\}_n$ is uniformly integrable, so is $\{\tilde{G}_n(\xi)\}_n$ is bounded in $L^1(R^d)$. Hence, $\{\Pi_n(\xi)\tilde{\Pi}_n^*(\xi)\}_n$ is uniformly integrable, hence by taking limit in the biorthogonal version of (3.31),

$$\left(\frac{1}{2\pi}\right)^d \int_{R^d} \Pi_n(\xi)\tilde{\Pi}_n^*(\xi)e^{\cdot\xi}d\xi = \delta_{0,k}I_m, \tag{5.30}$$

we get

$$F(\xi) = \sum_\alpha \hat{\varphi}(\xi + 2\pi\alpha)\hat{\tilde{\varphi}}^*(\xi + 2\pi\alpha) = I_m, \quad \text{a.e. } \xi, \tag{5.31}$$

which is the biorthogonality of $\{\varphi(x-k); \hat{\varphi}(x-\mathrm{k})\}$. As for the Riesz basis property of $\{\varphi_1(x - k), \ldots, \varphi_m(x - k)\}$, and of $\{\hat{\varphi}_1(x - \mathrm{k}), \ldots, \hat{\varphi}_m(x - \mathrm{k})\}$, we show that $F(\xi) = I_m$, together with a mild condition,

$$\Phi(\xi) + \tilde{\Phi}(\xi) \leqslant CI_m, \quad \text{a.e. } \xi, \tag{5.32}$$

will be sufficient. In fact, (5.32), together with $F(\xi) = I_m$, a.e. imply that for any $x \in C^d$, $|x| = 1$,

$$\begin{aligned} 1 = |x|^2 &= x^* \sum_{\alpha} \hat{\varphi}\hat{\tilde{\varphi}}^*(\xi + 2\pi\alpha)x \\ &= \sum_{i,j} \sum_{\alpha} \overline{x_i}\hat{\varphi}_i(\xi + 2\pi\alpha)\overline{\hat{\tilde{\varphi}}}_j(\xi + 2\pi\alpha)x_j \\ &= \sum_{\alpha} \sum_{i} \overline{x_i}\hat{\varphi}_i(\xi + 2\pi\alpha)\Big(\sum_{j} \overline{x_j}\hat{\tilde{\varphi}}_j(\xi + 2\pi\alpha)\Big)^- \\ &\leqslant \Big(\sum_{\alpha} \Big|\sum_{i} \overline{x_i}\hat{\varphi}_i(\xi + 2\pi\alpha)\Big|^2\Big)^{1/2}\Big(\sum_{\alpha} \Big|\sum_{j} \overline{x_j}\hat{\tilde{\varphi}}_j(\xi + 2\pi\alpha)\Big|^2\Big)^{1/2} \\ &= \Big(x^* \sum_{\alpha} \hat{\varphi}\hat{\varphi}^*(\xi + 2\pi\alpha)x\Big)^{1/2}\Big(x^* \sum_{\alpha} \hat{\tilde{\varphi}}\hat{\tilde{\varphi}}^*(\xi + 2\pi\alpha)x\Big)^{1/2} \\ &\leqslant C|x|(x^*\Phi(\xi)x)^{1/2}. \end{aligned}$$

Hence $\Phi(\xi) \geqslant CI_m$. Analogously, $\tilde{\Phi}(\xi) \geqslant CI_m$. Together with (5.32), $\{\varphi_1(x - k), \ldots, \varphi_m(x - k)\}_k$ is a Riesz basis of the space it generates, $\{\tilde{\varphi}_1, \ldots, \tilde{\varphi}_m(x - \mathrm{k})\}_k$ is also. But under the conditions assumed, (5.32) holds automatically, since $\Phi(\xi)$, $\tilde{\Phi}(\xi)$ are trigonometric polynomials, too. Thus, $\{V_j; \tilde{V}_j\}$ is a biorthogonal MRA(m) of $L^2(R^d)$, under the conditions (5.1), (5.2), (5.3), (5.4), and (5.5). The proof is finished.

Finally we consider the Riesz basis property of $\{\psi_{r,\mu,j,k}\}$ and of $\{\tilde{\psi}_{r,\mu,j,k}\}$, when $H_0(\xi)$, $\tilde{H}_0(\xi)$ are trigonometric polynomials.

LEMMA 5.1. *Assume that the family* $\{\psi_{r,\mu}\}$, $r = 1, \ldots, m$, $\mu \in E_d - \{0\}$, *satisfies*

$$|\hat{\psi}_{r,\mu}(\xi)| \leqslant C|\xi|^{\delta_1}(1 + |\xi|)^{-\delta_1-\delta_2}, \quad \text{a.e. } \xi, \tag{5.33}$$

and

$$\sum_{\alpha} |\hat{\psi}_{r,\mu}(\xi)|^{2-\delta_3} \leqslant C, \quad \text{a.e. } \xi, \tag{5.34}$$

where δ_1, δ_2, δ_3 *are positive constants. Then*

$$\sum_{r,\mu,j,k} |\langle f, \psi_{r,\mu,j,k}\rangle|^2 \leqslant C\|f\|_2^2 \quad \forall f \in L^2(R^d). \tag{5.35}$$

The proof has been given by Cohen and Daubechies [5].

THEOREM 5.4. *Let* $H_0(\xi)$, $\tilde{H}_0(\xi)$ *be trigonometric polynomials such that* (5.1), (5.2), (5.3), (5.4), *and* (5.5) *hold. Let the matrix-extensions* (5.9) *satisfy* (5.10) *and* (5.12).

Assume that all eigenvalues λ*'s of* P_{H_0}, *and all* $\tilde{\lambda}$*'s of* $P_{\tilde{H}_0}$ *restricted on* $\mathcal{P}$, *are in the unit circle. Then* $\{\psi_{r,\mu,j,k};\ \hat{\psi}_{r,\mu,j,k}\}$ *is a biorthogonal wavelet* (*Riesz*) *basis of* $L^2(R^d)$.

Proof. The assertions follow from Theorem 5.2, Theorem 5.3, Lemma 5.1, and the equalities

$$f = \sum \langle f, \tilde{\psi}_{r,\mu,j,k}\rangle \psi_{r,\mu,j,k} = \sum \langle f, \psi_{r,\mu,j,k}\rangle \cdot \tilde{\psi}_{r,\mu,j,k} \quad \forall f \in L^2(R^d).$$

Then the proof is finished.

6. ALGORITHMS AND EXAMPLES

For the construction of biorthogonal MRA(m) and biorthogonal wavelets, we summarize an algorithm based on Theorem 5.4 as follows. Obviously we can obtain the algorithm for constructing orthogonal MRA(m) and wavelets by letting $H_0(\xi) = \tilde{H}_0(\xi)$.

1. Select a couple $\{H_0(\xi);\ \tilde{H}_0(\xi)\}$ of trigonometric polynomial matrices, which should satisfy (5.1), (5.2), (5.3), (5.4), and (5.5).
2. Find a matrix-extension $\{(H_\mu(\xi + \nu\pi));\ (\tilde{H}_\mu(\xi + \nu\pi))\}$, which should satisfy (5.10) and (5.12);
3. Consider the action of P_{H_0}, $P_{\tilde{H}_0}$ on $\mathcal{P}_0$, where

$$\mathcal{P}_0 = \{ f(\xi) \colon (f(\xi))_{i,j} = \sum_{k \in \Delta_N} b_{i,j}(k) e^{-ik\cdot\xi}\},$$

and $\Delta_N = \prod_{i=1}^{d} [-N_i, N_i]$, $N_i = N_{i,+} - N_{i,-}$, and $\prod_{i=1}^{d} [N_{i,-}, N_{i,+}]$ is the coefficient support of the entries of $H_0(\xi)$ and $\tilde{H}_0(\xi)$; i.e., any entry of $H_0(\xi)$, $\tilde{H}_0(\xi)$ is of type

$$\sum_{k_1=N_{1,-}}^{N_{1,+}} \cdots \sum_{k_d=N_{d,-}}^{N_{d,+}} h_{i,j}(k) e^{-ik\cdot\xi}.$$

Then, check if all eigenvalues λ's of P_{H_0}, and all eigenvalues $\tilde{\lambda}$'s of $P_{\tilde{H}_0}$ are in the unit circle. If so, then $\{\hat{\varphi};\ \tilde{\varphi}\}$ defined by (5.13) generates a biorthogonal MRA(m), and $\{\psi_\mu;\ \tilde{\psi}_\mu\}$ defined by (5.14) generates a biorthogonal wavelet (Riesz) basis $\{\psi_{r,\mu,j,k};\ \tilde{\psi}_{r,\mu,j,k}\}$ of $L^2(R^d)$. If some of eigenvalues of P_{H_0} (when $\Phi(0) \geqslant I_m$ is assumed) is not in the unit circle, then $\{\varphi_1(x-k), \ldots, \varphi_m(x-k)\}_k$ is not a Riesz basis.

Step (3) of the algorithm is very easy to handle. So the construction is reduced to steps (1) and (2), which are difficult, even in the classical case $m = 1$, $d > 1$.

In order to calculate the eigen values of P_H, we need to know the matrix representation of operator P_H. Without loss of generality, let $M = I_m$, then for $F(\xi) \in \mathcal{P}(T^d, \mathcal{M}_m)$,

$$P_H F(\xi) = \sum_{\nu \in E_d} H\left(\frac{\xi}{2} + \nu\pi\right) F\left(\frac{\xi}{2} + \nu\pi\right) H^*\left(\frac{\xi}{2} + \nu\pi\right).$$

Let

$$H(\xi) = \sum_{k \in \Delta_1} D_k e^{-ik\xi}, \quad \text{where } \Delta_1 = \prod_{i=1}^{d} [N_i^-, N_i^+],$$

and

$$F(\xi) = \sum_{k \in \Delta_N} C_k e^{-ik\xi}.$$

Then

$$\begin{aligned} P_H F(\xi) &= \sum_{\nu \in E_d} \left(\sum_{k \in \Delta_1} D_k e^{-ik(\xi/2+\nu\pi)}\right)\left(\sum_{k \in \Delta_N} C_k e^{-ik(\xi/2+\nu\pi)}\right)\left(\sum_{k \in \Delta_1} D_k e^{-ik(\xi/2+\nu\pi)}\right) \\ &= \cdots \\ &= \sum_{m \in \Delta_N} 2^d \left(\sum_{k \in \Delta_N, 2m-k \in \Delta_N} \sum_{n \in \Delta_1, n-k \in \Delta_1} D_n C_{2m-k} D^*_{n-k}\right) e^{-im\xi} \\ &= \sum_{m \in \Delta_N} A_m e^{-im\xi}. \end{aligned}$$

Therefore P_H is the mapping: $\{C_l\}_{l \in \Delta_N} \mapsto \{A_m\}_{m \in \Delta_N}$, with

$$\begin{aligned} A_m &= 2^d \sum_{k \in \Delta_N, 2m-k \in \Delta_N} \sum_{n \in \Delta_1, n-k \in \Delta_1} D_n C_{2m-k} D^*_{n-k} \\ &= 2^d \sum_{l \in \Delta_N, 2m-l \in \Delta_N} \sum_{n \in \Delta_1, n-2m+l \in \Delta_1} D_n C_l D^*_{n-2m+l}. \end{aligned}$$

With respect to the basis $\{e^{-il\xi}\}_{l \in \Delta_N}$ of $\mathsf{P}(T^d, \mathsf{M}_m)$, we can write

$$P_H \sim (P_{m,l})_{\Delta_N \times \Delta_N},$$

where $P_{m,l}$ is the mapping: $C_l \mapsto \sum_{n \in \Delta_1, n-2m+l \in \Delta_1} 2^d D_n C_l D^*_{n-2m+l}$. Since $n \in \Delta_1$ and $n - 2m + l \in \Delta_1$ imply $2m - l \in \Delta_N$, we can be sure $P_{m,l} = 0$ when $2m - l \notin \Delta_N$. Now assume $2m - l \in \Delta_N$. With respect to the basis $\{E_{i,j}\}_{1 \leq i,j \leq m}$ of $\mathsf{M}_{m \times m}$, where $E_{i,j}$ is of, except for the (i, j)th-entry being 1, 0-entries, we can write

$$P_{m,l} \sim (P^{m,l}_{i,j})_{m^2 \times m^2},$$

with

$$P^{m,l}_{i,j} = 2^d \sum_{n \in \Delta_1, n-2m+l \in \Delta_1} d_{n,i_1,j_1} \overline{d}_{n-2m+l,i_2,j_2},$$

$$i = (i_1 - 1)m + i_2, \quad j = (j_1 - 1)m + j_2,$$

and d_{n,i_τ,j_τ} being the (i_τ, j_τ)th-entry of D_n, $\tau = 1, 2$. Let $B_k \in \mathcal{M}_{m^2}$ such that $B_k = 0$ when $k \notin \Delta$, and when $k \in \Delta$, $b_{k,i,j} = 2^{-d} \sum_{n\in\Delta_1} d_{n,i_1,j_1}\bar{d}_{n-k,i_2,j_2}$. Then we obtain

$$P_{m,l} = 2^d B_{2m-l}.$$

Now we take two examples to illustrate the algorithms. The first one is the orthogonal case and the second one is the biorthogonal case.

EXAMPLE 1. Take $a, b, c \in R - \{0\}$ such that

$$2a^2 + b^2 + c^2 = \tfrac{1}{2}.$$

Consider the function matrix

$$H_0(\xi) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}e^{-i\xi/2} & 0 \\ a - ae^{-i\xi/2} & b + ce^{-i\xi/2} \end{pmatrix}.$$

Obviously

$$H_0(0) = \begin{pmatrix} 1 & 0 \\ 0 & b + c \end{pmatrix}, \quad H_0(\pi) = \begin{pmatrix} 0 & 0 \\ 2a & b - c \end{pmatrix}.$$

Since $(b + c)^2 \leqslant 2(b^2 + c^2) < 1$, we deduce that one eigenvalue of $H(0)$ is 1, the modulus of another one is less than 1. It is easy to show that $H(\xi)$ satisfies (2.7) with $\epsilon = 1$. By Theorem 2.2, $\prod_1 (2^{-j})$ converges on any compact subset. It is also easy to verify that

$$H(\xi)H^*(\xi) + H(\xi + \pi)H^*(\xi + \pi) = I_2.$$

Now we calculate the eigenvalues of P_H by the above deduced result. In this time,

$$P_H = \begin{pmatrix} 2B_{-1} & 0 & 0 \\ 2B_1 & 2B_0 & 2B_{-1} \\ 0 & 0 & 2B_1 \end{pmatrix},$$

where

$$B_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2}(b + c) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(b + c) & 0 \\ 2a^2 & a(b - c) & a(b - c) & b^2 + c^2 \end{pmatrix}.$$

Hence $2B_0$ eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2(b^2 + c^2)$, and $\lambda_3 = \lambda_4 = b + c$. From $2a^2 + b^2 + c^2 = \frac{1}{2}$, and $a \neq 0$, we derive $|\lambda_i| < 1$ $(i = 2, 3, 4)$. By the same reason, we can deduce that the $2B_1$ and $2B_{-1}$ eigenvalues are all less than 1. Following our algorithm for orthogonal case, we get MRA(2).

Now we construct the associated wavelets. Select $p_i \in R$, $i = 1, 2, \ldots, 6$ such that

$$\begin{pmatrix} \sqrt{2}a & b & c \\ \sqrt{2}p_1 & p_2 & p_3 \\ \sqrt{2}p_4 & p_5 & p_6 \end{pmatrix}$$

is unitary, then

$$H_1(\xi) = \begin{pmatrix} p_1 & p_2 \\ p_4 & P_5 \end{pmatrix} + \begin{pmatrix} -p_1 & p_3 \\ -p_4 & P_6 \end{pmatrix} e^{-i\xi},$$

such that

$$\begin{pmatrix} H_0(\xi) & H_0(\xi + \pi) \\ H_1(\xi) & H_1(\xi + \pi) \end{pmatrix}$$

is unitary. It can be verified as follows. Let $H_1(\xi) = A_1 + A_2 e^{-i\xi}$, then

$$\begin{aligned} H_1(\xi)H_1^*(\xi) &= (A_1 + A_2 e^{-i\xi})(A_1^* + A_2^* e^{i\xi}) \\ &= (A_1A_1^* + A_2A_2^*) + A_2A_1^* e^{-i\xi} + A_1A_2^* e^{i\xi}. \end{aligned}$$

Hence

$$H_1(\xi)H_1^*(\xi) + H_1(\xi + \pi)H_1^*(\xi + \pi) = A_1A_1^* + A_2A_2^*.$$

From

$$A_1A_1^* + A_2A_2^* = \begin{pmatrix} p_1^2 + p_2^2 + P_1^2 + p_3^2 & p_1p_4 + p_2p_5 + p_1p_4 + p_3p_6 \\ p_1p_4 + p_2p_5 + p_1p_4 + p_3p_6 & p_4^2 + p_5^2 + p_4^2 + p_6^2 \end{pmatrix},$$

we know $A_1A_1^* + A_2A_2^* = I_2$. By now we have shown:

$$H_1(\xi)H_1^*(\xi) + H_1(\xi + \pi)H_1^*(\xi + \pi) = I_2.$$

By similar calculation, we can also show:

$$H_0(\xi)H_1^*(\xi) + H_0(\xi + \pi)H_1^*(\xi + \pi) = 0.$$

Finally to create the orthogonal wavelet is just to follow the algorithm routinely.

Since the calculation is very complicated, now we just give a simple example to illustrate the algorithm for biorthogonal case.

EXAMPLE 2. Let $\{V_j\}$, $\{\tilde{V}_j\}$ be two MRA of $L^2(R)$ with filter functions $m_0(\xi)$, $\tilde{m}_0(\xi)$ (in $C(\Pi)$) satisfying $\Sigma_{\nu=0}\ m_0(\xi + \nu\pi)\tilde{m}_0(\xi + \nu\pi) = 1$. Define $m_1(\xi) = e^{-i\xi}\tilde{m}_0(\xi + \pi)$, $\tilde{m}_1(\xi) = e^{-i\xi}\overline{m_0}(\xi + \pi)$. Denote $\hat{\varphi}(\xi) = (\hat{\varphi}_0(\xi), \hat{\varphi}_1(\xi))^t$, $\tilde{\varphi}(\xi) = (\tilde{\varphi}_0(\xi), \tilde{\varphi}_1(\xi))^t$, with $\hat{\varphi}_i(\xi) = m_i(\xi/2)\hat{\varphi}_i(\xi/2)$, $i = 0, 1$, $\hat{\varphi}_0(\xi) = \prod_1^\infty m_0(2^{-j}\xi)$ and similarly for $\tilde{\varphi}_i(\xi)$. Notice that

$$\begin{pmatrix} m_0(\xi) & m_0(\xi + \pi) \\ m_1(\xi) & m_1(\xi + \pi) \end{pmatrix} \begin{pmatrix} \overline{\tilde{m}_0}(\xi) & \overline{\tilde{m}_1}(\xi) \\ \overline{\tilde{m}_0}(\xi + \pi) & \overline{\tilde{m}_1}(\xi + \pi) \end{pmatrix} = I_2 \quad \forall\xi. \tag{6.1}$$

Notice that $\{\varphi_0(x - k), \varphi_1(x - k)\}_k$ generates V_1. Similarly for $\tilde{V}_1$ (see, for example, [26]). Consider $\{V_j, \tilde{V}_j\}$, with $V_0 = V_1$, $V_j = 2^j V_0$, and similarly for $\tilde{V}_j$. Now we do some calculations. Since $\hat{\varphi}_0(\xi) = m_0(\xi/2)\hat{\varphi}_0(\xi/2)$, $\hat{\varphi}_1(\xi) = m_1(\xi/2)\hat{\varphi}_0(\xi/2)$, similarly for $\tilde{\varphi}$, we have

$$H_0(\xi) = \begin{pmatrix} m_0(\xi) & 0 \\ m_1(\xi) & 0 \end{pmatrix}, \quad \tilde{H}_0(\xi) = \begin{pmatrix} \tilde{m}_0(0) & 0 \\ \tilde{m}_1(0) & 0 \end{pmatrix}. \tag{6.2}$$

Since

$$H_0(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \tilde{H}_0(0), \quad H_0(\pi) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \tilde{H}_0(\pi),$$

we see that

$$H_0^*(\nu\pi)e_1 = \delta_{0,\nu}e_1 = \tilde{H}_0^*(\nu\pi), \quad \nu = 0, 1. \tag{6.3}$$

And from (6.1) we have

$$\sum_\nu H_0(\xi + \nu\pi)\tilde{H}_0^*(\xi + \nu\pi) = I_2 \quad \forall\xi. \tag{6.4}$$

Notice that in this case, we have always

$$\Phi(0) = I_2 = \tilde{\Phi}(0), \tag{6.5}$$

which follows from the calculations

$$\sum_\alpha |\hat{\varphi}_0(2\pi\alpha)|^2 = 1 = \sum_\alpha |\hat{\varphi}_1(2\pi\alpha)|^2, \quad \sum_\alpha \hat{\varphi}_0(2\pi\alpha)\hat{\varphi}_1(2\pi\alpha) = 0.$$

Assume that both $m_0(\xi)$, $\tilde{m}_0(\xi)$ are trigometric polynomials, and

$$|m_0(\xi)| = 1 + O(|\xi|^\epsilon). \tag{6.6}$$

Equations (6.2), (6.3), (6.4), (6.5), (6.6) show that $\{H_0(\xi); \tilde{H}_0(\xi)\}$ satisfies the step

(1) in the preceding algorithm, so we can get a biorthogonal MRA(2) by making use of the eigenvalue estimate of P_{H_0} and $P_{\tilde{H}_0}$, indicated in step (3).

What are the related wavelets? Or, equivalently, what are the possible $\{H_1(\xi); \tilde{H}_1(\xi)\}$? The following simple setting of $H_1(\xi)$ and $\tilde{H}_1(\xi)$,

$$H_1(\xi) = \begin{pmatrix} 0 & a(\xi) \\ 0 & b(\xi) \end{pmatrix}, \quad \tilde{H}_1(\xi) = \begin{pmatrix} 0 & \tilde{a}(\xi) \\ 0 & \tilde{b}(\xi) \end{pmatrix}, \tag{6.7}$$

will do, where $a(\xi)$, $b(\xi)$, $\tilde{a}(\xi)$, $\tilde{b}(\xi)$ are in $C(\Pi)$ and satisfy

$$\sum_\nu a(\xi + \nu\pi)\bar{\tilde{a}}(\xi + \nu\pi) = 1 = \sum_\nu b(\xi + \nu\pi)\bar{\tilde{b}}(\xi + \nu\pi) \quad \forall\xi, \tag{6.8}$$

$$\sum_\nu a(\xi + \nu\pi)\bar{\tilde{b}}(\xi + \nu\pi) = 0 = \sum_\nu b(\xi + \nu\pi)\bar{\tilde{a}}(\xi + \nu\pi) \quad \forall\xi. \tag{6.9}$$

In fact, with such setting of $(H_\nu(\xi + \nu\pi))_{\mu,\nu}$ and $(\tilde{H}_\nu(\xi + \nu\pi))_{\mu,\nu}$, (5.10) and (5.12) hold, provided $m_1(\xi)$, $\tilde{m}_1(\xi) = O(|\xi|^\epsilon)$. When $a(\xi) = 2^{-1/2} = \tilde{a}(\xi)$, $b(\xi) = 2^{-1/2}e^{i\xi} = \tilde{b}(\xi)$, we have

$$\hat{\psi}_0(\xi) = 2^{-1/2}\hat{\varphi}_1\left(\frac{\xi}{2}\right), \quad \hat{\psi}_1(\xi) = 2^{-1/2}e^{-i(\xi/2)}\hat{\varphi}_1\left(\frac{\xi}{2}\right),$$

$$\psi_0(x) = 2^{1/2}\varphi_1(2x), \quad \psi_1(x) = 2^{1/2}\varphi_1(2x - 1).$$

This gives the trivial basis $\{2^{1/2}\psi(2x - k)\}_k$ of $V_1 \ominus V_0 = V_2 \ominus V_1 = W_1$, which is nothing but the classical one.

ACKNOWLEDGMENTS

The authors greatly appreciate the referees for their helpful suggestions. They also thank Dr. Q. Mo for his partial contribution in the last section.

REFERENCES

1. C. de Boor, R. DeVore, and A. Ron, On the construction of multivariate (pre)wavelets, *Const. Appr.* **9** (1993), 123–166.
2. C. de Boor, R. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L^2(R^d)$, *J. Funct. Anal.* **119** (1994), 37–78.
3. C. K. Chui and J. Lian, A study on orthonormal multi-wavelets, Appl. Numer. Math. **20** (1996), 273–298.
4. A. Cohen, Ondelettes, analyses multirésolutions et filtres miroirs en quadrature, *Ann. Inst. H. Poincaré Anal. non-linéaire* **7** (1990), 439–459.
5. A. Cohen and I. Daubechies, A stability criterion for biorthogonal wavelet bases and their related subband coding schemes, *Duke Math. J.* **68** (1992), 313–335.
6. A. Cohen and I. Daubechies, Non-separable bidimensional wavelet Bases, *Rev. Mat. Iberoamericana* **9** (1993), 51–137.

7. A. Cohen, I. Daubechies, and J. C. Feauveau, Biorthogonal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **45** (1992), 485–560.
8. A. Cohen, N. Dyn, and D. Levin, Stability and inter-dependence of matrix subdivision, *in* ‘‘Advanced Topics in Multivariate Approximation’’ (F. F. Fontanelle, K. Jetter, and L. Schumaker, Eds.), pp. 1–13, World Science, Singapore, 1996.
9. A. Cohen, I. Daubechies, and G. Plonka, Regularity of refinable function vectors. [preprint]
10. J. P. Conze and A. Raugi, Fonctions harmoniques pour un operatéur de transition et applications, *Bull. Soc. Math. France* **118** (1990), 273–310.
11. W. Dahmen and C. A. Micchelli, Biorthogonal wavelet expensions, *Constr. Approx.* **13,** No. 3 (1997), 293–328.
12. I. Daubechies, Ten lectures on wavelets, *in* ‘‘CBMS-Conference Lecture Notes, Vol. 61,’’ SIAM, Philadelphia, 1992.
13. I. Daubechies and J. Lagarias, Two-scale difference equations. I. Existenxe and global regularity of solutions, *SIAM J. Math. Anal.* **22** (1991), 1388–1410. [II. Local regularity, infinite products of matrices and fractals, *SIAM J. Math. Anal.* **23** (1992), 1031–1079]
14. G. Donovan, J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Construction of orthogonal wavelets using fractal interpolations, *SIAM J. Math. Anal.* [to appear]
15. J. S. Geronimo, D. P. Hardin, and P. R. Massopust, Fractal functions and wavelet expansions based on several scaling functions, *J. Approx. Theory* **78** (1994), 373–401.
16. T. N. T. Goodman and J. Lee, Wavelets of multiplicity r, *Trans. Amer. Math. Soc.* **342** (1994), 307–324.
17. T. N. T. Goodman, S. L. Lee, and W. S. Tang, Wavelets in wandering subspaces, *Trans. Amer. Math. Soc.* **338** (1993), 639–654.
18. D. P. Hardin, B. Kessler, and P. R. Massopust, Multiresolution analysis based on fractal functions, *J. Approx. Theory* **71** (1992), 104–120.
19. C. Heil and D. Colella, Matrix refinement equation: existence and uniqueness, *J. Fourier Anal. Appl.* **2,** No. 4 (1996), 363–377.
20. L. Hérve, Régularité et conditions de bases de Riesz pour les fonctions d’échelle, *C. R. Acad. Sci.* Paris Sér. I **315** (1992), 1029–1032.
21. L. Hérve, Multiresolution analysis of multiplicity d; Applications to dyadic interpolation, *Appl. Comput. Harmon. Anal.* **1** (1994), 299–315.
22. R. Q. Jia and Z. W. Shen, Multiresolution and wavelets, *Proc. Edinburgh Math. Soc.* **37** (1994), 271–300.
23. W. M. Lawton, Necessary and sufficient conditions for constructing orthonormal wavelet bases, *J. Math. Phys.* **32** (1991), 57–61.
24. W. M. Lawton, L. Lee, and L. Shen, Stability and orthonormality of multivariate refinable functions. [preprint]
25. R. Long, ‘‘Hidimensional Wavelet Analysis’’ Advanced Educational Press, Beijing, 1995. [Chinese]
26. R. Long and D. Chen, Biorthogonal wavelet bases on R^d, *Appl. Comput. Harmon. Anal.* **2** (1995), 230–242.
27. R. Long and W. Chen, Wavelet basis packet and wavelet frame packet, *J. Fourier Anal. Appl.* **3,** No. 3 (1997).
28. R. Long and Q. Mo, L^2-convergence of vector cascade algorithm. [preprint]
29. S. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$, *Trans. Amer. Math. Soc.* **315,** No. 1 (1989), 69–88.
30. G. Strang and V. Strela, Orthogonal multiwavelets with vanishing moments, *J. Opt. Eng.* **33** (1994), 2104–2107.
31. I. F. Villemoes, Energy moments in time and frequency for two-scale difference equation solutions and wavelets, *SIAM J. Math. Anal.* **23** (1992), 108–115.