Time Average MSE Analysis for the First Order Sigma-Delta Modulator with the Inputs of Bandlimited Signals

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Abstract—Based on experiments and numerical simulation, it has been widely believed that the time average mean square error in the first order sigma-delta modulator with input of bandlimited signals decays like $O(\lambda^{-2})$ as the sampling ratio $\lambda$ goes to infinity. This conjecture remains as an open problem for many years. Combining tools from number theory, harmonic analysis, real analysis and complex analysis, this paper shows that the conjecture holds in some reasonable sense.

I. INTRODUCTION

Converting an analog signal into a digital signal (A/D conversion) consists of two steps: sampling and quantization. According to sampling theorems [3], [4], [7], [16], the loss in the sampling step is reversible. However, the quantization step usually introduces an irreversible loss of information. The loss in the quantization step is called the quantization error. Recent research shows that the accuracy of A/D conversion can be improved by refining the resolution of sampling as well as quantization [3], [6]. Due to the expensive cost in building a high resolution quantizer, high accuracy of A/D conversion is usually achieved by refining the resolution of sampling. Among many schemes, the sigma-delta ($\Sigma\Delta$) modulator provides a promising architecture for high accuracy A/D conversion because it is robust against circuit imperfections and hence is amenable to LSI and VLSI architecture for high accuracy A/D conversion.

In the literature, the input signal $x$ to a $\Sigma\Delta$ modulator is modelled to be a bandlimited function taking values in the interval $[0, 1]$ or $[-1, 1]$ for the convenience of discussion [7], [9], [10], [11], [12]. For the same reason, in this paper, we consider the signal $x$ to be a bandlimited function taking values in the interval $[1, 2]$ instead. But these assumptions are essentially equivalent. In this paper, we work with the space tempered distributions in the usual way [8]. Without loss of generality, in this paper, we consider the bandlimited signals in the space $L^\infty(\mathbb{R})$ so that some important signals, such as sinusoidal signals, can be included in our discussion. In other words, we define the signal class to be

$$B_0 := \{ x \in L^\infty(\mathbb{R}) \mid \hat{x} \text{ has support contained in } [-\Omega, \Omega] \},$$

where $\hat{x}$ denotes the Fourier transform of $x$ which is defined by $\hat{x}(\xi) := \int_{-\infty}^{\infty} x(t)e^{-i\xi t}dt$ for $x \in L^1(\mathbb{R})$ and extended to the tempered distributions in the usual way [8]. Without loss of generality, in this paper we work with the space $B_0$ and all the results in this paper can be easily extended to arbitrary bandwidth $\Omega$ by rescaling. For $\lambda > 1$, let $\varphi$ be a function in the Schwartz class such that $\varphi = 1$ on the interval $[-\pi, \pi]$ and $\varphi$ has support contained in the interval $[-\lambda \pi, \lambda \pi]$. Then the following sampling formula [9], [10], [14] holds

$$x(t) = \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} x\left(\frac{n}{\lambda}\right) \varphi(t - \frac{n}{\lambda}), \quad x \in B_0.$$  \hspace{1cm} (I.1)

This $\varphi$ is called an admissible reconstruction filter, the quantity $\lambda$ in the above sampling formula is called the sampling ratio, and $\{x(n/\lambda)\}_{n \in \mathbb{Z}}$ are called the samples of the input signal $x$.

For $t \in \mathbb{R}$, we denote by $\lfloor t \rfloor$ the largest integer that is not greater than $t$ and denote $(t) := t - \lfloor t \rfloor \in [0, 1)$. The $\Sigma\Delta$ modulator uses the samples $\{x(n/\lambda)\}_{n \in \mathbb{Z}}$ as inputs to generate the binary digital signal $\{q_\lambda(n)\}_{n \in \mathbb{Z}}$ as follows:

$$X_\lambda(n) := \sum_{m=1}^{n} x\left(\frac{m}{\lambda}\right), \quad (I.2)$$

$$Q_\lambda(n) := \lfloor X_\lambda(n) \rfloor, \quad (I.3)$$

$$q_\lambda(n) := Q_\lambda(n) - Q_\lambda(n-1), \quad (I.4)$$

where $X_\lambda(0) := 0$ and $X_\lambda(n) := -\sum_{m=n+1}^{0} x(m/\lambda)$ for $n < 0$. It is easy to see that $X_\lambda(n) - X_\lambda(n-1) = x(n/\lambda)$ for all $n \in \mathbb{Z}$. Since $x$ takes value in the interval $[1, 2]$, it is easy to check that $q_\lambda(n)$ takes the binary value 1 or 2. Note that the equations (I.2) and (I.4) correspond to $\Sigma$ and $\Delta$, respectively; hence giving the name of the modulator. Since $X_\lambda$ and $Q_\lambda$ will accumulate into huge numbers as time elapses, neither can be calculated in a circuit. Thus one introduces the auxiliary variable $u_\lambda := X_\lambda - Q_\lambda = \langle X_\lambda \rangle$. Then $u_\lambda$ satisfies the recursive relation:

$$u_\lambda(n) - u_\lambda(n-1) = x\left(\frac{n}{\lambda}\right) - q_\lambda(n).$$  \hspace{1cm} (I.5)

Since $u_\lambda(n) \in [0, 1]$, from (I.5) one has the relation $q_\lambda(n) = [x(n/\lambda) + u_\lambda(n-1)]$. Using the auxiliary variable $u_\lambda$, now we can translate the procedure in (I.2)-(I.4) into the following equivalent procedure:

$$u_\lambda(n) := u_\lambda(n-1) + x\left(\frac{n}{\lambda}\right) - q_\lambda(n) \quad \text{with} \quad u_\lambda(0) := 0.$$  \hspace{1cm} (I.6)

$$q_\lambda(n) := \begin{cases} 1, & \text{if} \quad x\left(\frac{n}{\lambda}\right) + u_\lambda(n-1) < 2; \\ 2, & \text{if} \quad x\left(\frac{n}{\lambda}\right) + u_\lambda(n-1) \geq 2. \end{cases}$$  \hspace{1cm} (I.7)

The simplest $\Sigma\Delta$ modulator is shown in Figure 1 and consists of a discrete time integrator and a binary quantizer inside a single feedback loop [7], [12], [18].

Reconstructing the analog signal from the digital sequence $\{q_\lambda(n)\}_{n \in \mathbb{Z}}$ by the sampling formula in (I.1), one obtains the
reconstructed signal

\[ \tilde{x}(t) := \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} q_n(n) \phi \left( t - \frac{n}{\lambda} \right). \]

Then the quantization error is defined to be the difference \( e_\lambda(x, t) := x(t) - \tilde{x}(t) \). The basic estimate given in [7], [9] for the quantization error is

\[ \left\| e_\lambda(x, \cdot) \right\|_\infty \lesssim \| \lambda^{\nu} \|^{-1}. \]

In practice, however, one observes a much better decay behaviour than that of this basic estimate. In particular, through experiments and numerical simulation, it is commonly believed that

\[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e_\lambda(x, t) \right|^2 dt \lesssim C \lambda^{-3}. \]

with \( C \) independent of the input signal \( x \).

However, it is turn out to be very hard to improve the basic estimate [7]. For very restricted classes of signals \( x \), the conjecture in (I.8) has been proved in some sense. In particular, Gray [11] showed that if \( x_n = a \) for \( a \in [-1, 1] \), then

\[ \int_{-1}^{1} \left[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e_\lambda(x_n, t) \right|^2 dt \right] da \lesssim C \lambda^{-3}. \]

Gray’s result was later extended by Gray, Chou and Wong [12] to the case where the input signal \( x \) is sinusoidal; that is, \( x(t) = a \sin b t \), with \( |b| < \pi \).

For general bandlimited signals, to our best knowledge, there was little progress on this conjecture until the recent interesting work of G"unt"urk [9], [10], who proved, by a combination of tools from number theory and harmonic analysis, that if \( x \in B_a \) takes values in the interval \([0, 1] \), then for any small \( \varepsilon > 0 \) and all \( t \) satisfying \( x'(t) \neq 0 \), there is a constant \( C_{\varepsilon, x'}(t) > 0 \) such that

\[ \left| e_\lambda(x, t) \right|^2 \leq C_{\varepsilon, x'}(t) \lambda^{-8/3 + \varepsilon}, \]

where the constant \( C_{\varepsilon, x'}(t) \) depends on \( \varepsilon \) and the local value \( x'(t) \).

Our work in this paper has been greatly motivated and inspired by the recent interesting work in Daubechies and DeVore [7] and G"unt"urk [9], [10]. Our objective in this paper is to try to prove the conjecture in (I.8). Using tools from number theory, harmonic analysis, real analysis and complex analysis, we shall prove the conjecture in the sense that the time average MSE decays like \( O(\lambda^{-3 + \varepsilon}) \) for any small \( \varepsilon > 0 \) as the sampling ratio \( \lambda \) approaches infinity. Furthermore, if the signals are restricted to some special classes including the periodic signals, our result is very much close to the original conjecture in (I.8). See the next section for details of the main results in this paper.

II. MAIN RESULTS

We now present the main results in this paper. Since the conjecture in (I.8) has been confirmed by Gray [11] and G"unt"urk [9], [10] for a constant input, in the following we only deal with a non-constant signal \( x \in B_a \).

**Theorem 1:** Suppose that \( x \in B_a \) takes values in the interval \([1, 2] \). Then for any large \( T > 0 \) and any small \( \varepsilon > 0 \), there are two positive constants \( \lambda_{\varepsilon, T} \) and \( C_{\varepsilon, x, T} \) such that

\[ \frac{1}{2T} \int_{-T}^{T} \left| e_\lambda(x, t) \right|^2 dt \leq C_{\varepsilon, x, T} \lambda^{-3 + \varepsilon} \quad \forall \lambda > \lambda_{\varepsilon, T}, \]

where \( C_{\varepsilon, x, T} \) depends on \( \varepsilon \), \( T \) and the property of the signal \( x \).

Theorem 1 implies that, considering the input signal \( x \) on any finite time interval \([-T, T] \), the decay of the average square error behaves like \( O(\lambda^{-3 + \varepsilon}) \) for any small \( \varepsilon > 0 \). Since \( \varepsilon \) can be arbitrarily small, Theorem 1 is close to the conjecture in (I.8) on a finite time interval. Since in experiments and numerical simulation one always deals with signals on a finite time interval, Theorem 1 explains well the observations made in experiments.

The constant \( C_{\varepsilon, x, T} \) in Theorem 1 is determined by the behaviour of the signal \( x \) in the interval \((-T - \tau, T + \tau) \) for any fixed \( \tau > 0 \). Suppose that the increasing sequence \( \{\alpha_n\} \) are all the distinct zeros of \( x' \) on \( \mathbb{R} \), where \( \lambda \) is a set of consecutive integers. Suppose also that \( \{\alpha_n\} \) are all the distinct zeros of \( x' \) on \([-T, T] \). If there is a zero of \( x' \) in the open interval \((-T - \tau, T) \), we set \( T_0 := \alpha_{\lambda(T)_-} \) and \( \sigma_0 := -M(T) - 1 \), otherwise \( T_0 := -T \) and \( \sigma_0 := -M(T) \); if there is a zero of \( x' \) in the open interval \((T, T + \tau) \), we set \( T_0 := \alpha_{\lambda(T)_+} + 1 \) and \( \sigma_0 := N(T) + 1 \), otherwise \( T_0 := T \) and \( \sigma_0 := N(T) \). Let \( \alpha_0 \) denote the order of the zero \( \alpha_0 \) of the analytic function \( x' \) for all \( v \in \Lambda \), and \( \tau_0 := \sup_{m \in \Lambda} \varepsilon \tau_v \). Let \( m_\tau := \min \{ m \in \mathbb{N} : |x'(\alpha_m)| > \varepsilon \} \). Then \( m_\tau > 0 \). For any positive small \( d < \tau \), let \( \tau_0 \) be disjoint to each other for all \( v = \sigma_0, \ldots, \beta_0 \), let

\[ c_\varepsilon(d) := \inf_{v \in \Lambda} \inf_{t \in V(\alpha_v, d)} \left| x'(t) \right|. \]

Then \( c_\varepsilon(d) > 0 \). As a matter of fact, the constant \( C_{\varepsilon, x, T} \) in Theorem 1 is determined by the quantities \( \varepsilon, \tau, m_\tau \) and \( c_\varepsilon(d) \). Naturally, if we put some conditions on the quantities \( \tau_0, m_\tau \) and \( c_\varepsilon(d) \), the result in Theorem 1 can be improved and we have the following theorem.

**Theorem 2:** Suppose that \( x \in B_a \) takes values in the interval \([1, 2] \) and \( \sup_{|\alpha| \leq \Lambda} \varepsilon \tau_v < \infty \). If there is a real number \( d > 0 \) such that \( V(\alpha_v, d) \) is disjoint to each other for all \( v \in \Lambda \), then for any small \( \varepsilon > 0 \), there exist two positive constants \( \lambda_{\varepsilon} \) and \( C_{\varepsilon, x} \) such that

\[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e_\lambda(x, t) \right|^2 dt \leq C_{\varepsilon, x} \lambda^{-\varepsilon} \quad \forall \lambda > \lambda_{\varepsilon}, \]

where the constant \( C_{\varepsilon, x} \) is independent of \( T \) and \( \lambda \),

For any periodic signal, the quantities \( \tau_0, m_\tau \) and \( c_\varepsilon(d) \) are completely determined by its behaviour in one period. Therefore, \( \tau_0, m_\tau \) and \( c_\varepsilon(d) \) are independent of \( T \). Hence we have the following result.

**Corollary 1:** Suppose that \( x \in B_a \) is periodic and takes values in the interval \([1, 2] \). Then for any small \( \varepsilon > 0 \), there exist two positive constants \( \lambda_{\varepsilon} \) and \( C_{\varepsilon, x} \) such that

\[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| e_\lambda(x, t) \right|^2 dt \leq C_{\varepsilon, x} \lambda^{-\varepsilon} \quad \forall \lambda > \lambda_{\varepsilon}. \]

The sinusoidal signals considered by Gray et al [12] are obviously covered by Corollary 1. However, the constant \( C_{\varepsilon, x} \) in Theorem 2 and Corollary 1 is not independent of the input \( x \). We guess that such an absolute constant may depend on the statistical properties of the signals in \( B_a \). In Gray’s estimate, he got the absolute constant since he took the integration over all possible signals \( x_n = a \sin bt \) for \( a \in [-1, 1] \). It could be possible to define a measure in \( B_a \) and take integration of the average square error with respect to the measure to obtain an absolute constant.

III. PROOF OF THEOREM 1

In this section, we will present the proof of Theorem 1. Since Theorem 2 and Corollary 1 are immediate consequences of our proof
of Theorem 1, we will omit their proofs. In this section, we always use $C$ to stand for a general absolute constant.

Before proceeding further, we need some auxiliary results. First, we review some important facts about bandlimited functions in [19].

A bandlimited function $x \in B_0$ is the restriction to $\mathbb{R}$ of an entire function of exponential type $\Omega$ [19, page 85]; that is, the function defined by

$$x(z) = \frac{1}{\pi} \int_{-\Omega}^{\Omega} e^{i\xi z} \hat{x}(\xi) d\xi, \quad z \in \mathbb{C},$$

is an entire function and satisfies $|x(z)| \leq A e^{\Omega|z|}$ for all $z \in \mathbb{C}$ with some constant $A > 0$.

Bernstein’s inequality [19, page 88]: If $x \in B_0$, then $\|x^{(r)}\|_{\infty} \leq \Omega^r \|x\|_{\infty}$.

For a function $x$ of complex variable, $z_0$ is called a zero of $x$ if $x(z_0) = 0$. An interesting result about the zeros of an entire function of exponential type which plays an important role in this paper is as follows.

Proposition 1: (see [19, page 52]) If $x(z)$ is a nonzero entire function of exponential type $\Omega$, then

$$\lim_{T \to \infty} \sup_{t \in [0, T]} \frac{N(T)}{T} \leq \frac{2\Omega}{\ln 2},$$

where $N(T)$ is the number of zeros of $x$ inside the disc with radius $T$ and center at the origin.

The following fundamental estimate in number theory was used in Güntürk’s work [9, 10] and will be useful in this paper.

Lemma 1: (Koksma-Erdős-Turán [13, 15]) Let $f$ be a function of bounded variation on $[0, 1]$. For any finite sequence of points $u_1, \ldots, u_n$ in $[0, 1]$,

$$\left| \frac{1}{n} \sum_{k=1}^{n} f(u_k) - \int_{0}^{1} f(u) du \right| \leq CV_0^1(f) \inf_{K \geq 1} \left( \frac{1}{K} + \sum_{k=1}^{K} \left| \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i u_k k} \right| \right),$$

where $V_0^1(f)$ is the total variation of $f$ on $[0, 1]$.

We also need an estimate for oscillatory integrals in harmonic analysis.

Lemma 2: (van der Corput [17, page 332]) Suppose that $\beta$ is a real-valued function on the interval $(a, b)$ such that for a positive integer $r$, $|\beta^{(r)}(t)| \geq \mu$ for all $t \in (a, b)$. Then

$$\left| \int_{a}^{b} e^{i\beta(t)} dt \right| \leq C_{r}\mu^{-1/r},$$

holds when:

(i) $r \geq 2$

(ii) $r = 1$ and $\beta$ is monotonic in $(a, b)$.

The constant $C_r$ depends only on $r$.

The following result is an extension of an interesting result established by Güntürk [9, 10], which will be useful in this paper.

Lemma 3: (Güntürk [9, 10]) The sequence $X_{\lambda}$ can be extended to an analytic function, which we shall denote by $X_{\lambda}$ as well. Moreover, for any positive integer $r$,

$$X_{\lambda}^{(r)} = \frac{1}{\lambda^{r-1}} \left( \frac{\lambda}{x(\lambda)} - 1 \right) + R_{r, \lambda},$$

and

$$\|R_{r, \lambda}\|_{\infty} \leq \frac{|\lambda|^{r-1} e^{\pi/\lambda} \|x\|_{\infty}}{1 - \pi/\lambda}.$$

Now we present the proof of Theorem 1. For any $t \in \mathbb{R}$, define

$$N_{\lambda}(t) := \lfloor \lambda t \rfloor, \quad t_{\lambda} := \frac{N_{\lambda}(t)}{\lambda} \quad \text{and} \quad \delta_{\lambda}(t) := t - t_{\lambda} = \frac{\lfloor \lambda t \rfloor}{\lambda}.$$

Let $\varphi$ be a function in the Schwartz class such that $\hat{\varphi} = 1$ on $[-\pi, \pi]$ and has support contained in $[-2\pi, 2\pi]$. Therefore, there are constants $C_{r, \lambda} > 0$ such that for all $N \geq 0$ and $r \geq 0$,

$$|\varphi^{(r)}(t)| \leq C_{r, \lambda} / (1 + |t\lambda|^r) \quad \forall \ t \in \mathbb{R}.$$

For a small $\epsilon > 0$, set $\Omega_{\lambda} := \lambda^{r/8}$ and define $\varphi_{\lambda}$ by

$$\varphi_{\lambda}(t) := \Omega_{\lambda} \varphi(\Omega_{\lambda} t).$$

Then $\varphi_{\lambda}(z) = \varphi(z / \Omega_{\lambda})$. Since $\varphi_{\lambda} = 1$ on $[-\Omega_{\lambda} \pi, \Omega_{\lambda} \pi] \subset [-\pi, \pi]$ and $\varphi_{\lambda}$ has support contained in $[-2\Omega_{\lambda} \pi, 2\Omega_{\lambda} \pi] \subset [-\lambda \pi, \lambda \pi]$ for $\lambda > 4$, $\varphi_{\lambda}$ is a family of admissible reconstruction filters. Define the difference operator $\Delta_{\lambda}$ by $\Delta_{\lambda} \varphi := \varphi - \varphi(-\cdot - s)$ and denote $\Delta := \Delta_{\lambda}$. From the sampling formula in (1.1) and the definition of quantization error, by $t = t_{\lambda} + \delta_{\lambda}(t)$ and $\Delta u_{\lambda}(n) = x(n/\lambda) - q_{\lambda}(n, t)$, we have

$$e_{\lambda}(x, t) = \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} (x(n/\lambda) - q_{\lambda}(n)) \varphi_{\lambda} \left( t - \frac{n}{\lambda} \right)$$

$$= \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} \Delta u_{\lambda}(n) \varphi_{\lambda} \left( t - \frac{n}{\lambda} \right)$$

$$= \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} \Delta u_{\lambda}(n + N_{\lambda}(t)) \Delta \varphi_{\lambda} \left( \delta_{\lambda}(t) - \frac{\lambda}{\lambda} \right).$$

Since $u_{\lambda} \in (0, 1)$, we deduce that

$$h_{\lambda}(t) := \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} u_{\lambda}(n + N_{\lambda}(t))$$

satisfies

$$|h_{\lambda}(t)| \leq \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} |\Delta \varphi_{\lambda} | \left( \delta_{\lambda}(t) - \frac{n}{\lambda} \right) - \Delta \varphi_{\lambda} \left( -\frac{n}{\lambda} \right)$$

$$\leq \frac{1}{\lambda} \sum_{n \in \mathbb{Z}} \int_{-n/\lambda}^{n/\lambda} |\Delta \varphi_{\lambda}(s)| \|\varphi_{\lambda}^{(r)}(s)\| ds$$

$$\leq \frac{1}{\lambda} \int_{-\infty}^{\infty} ds \int_{-1/\lambda}^{1/\lambda} |\varphi_{\lambda}^{(r)}(s)| ds$$

$$= \lambda^{-2 + 1/4} \|\varphi_{\lambda}^{(r)}\|_{2}.$$
Then $\varepsilon_{\lambda}(\lambda t + n) - 1/2 = \Delta U_{\lambda}(n, t)$. Therefore,
\[
e_{\lambda}(x, t) = \sum_{n=0}^{\infty} \Delta U_{\lambda}(n, t) \Delta_{\lambda} \varphi_{\lambda}(\frac{n x}{\lambda}) + h_{\lambda}(t)
\]
\[
= \frac{1}{\lambda} \sum_{n=0}^{\infty} U_{\lambda}(n, t) \Delta_{\lambda}^{2} \varphi_{\lambda}(\frac{n x}{\lambda}) + h_{\lambda}(t)
\]
\[
= \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda} \sum_{n=0}^{\infty} + \frac{1}{\lambda} \sum_{n=0}^{\infty} U_{\lambda}(n, t) \Delta_{\lambda}^{2} \varphi_{\lambda}(\frac{n x}{\lambda}) + h_{\lambda}(t)
\]
\[
= : e_{\lambda}(x, t) + 0 + e_{\lambda}^*(x, t) + h_{\lambda}(t).
\]
In order to estimate $e_{\lambda}^*(x, t)$, it suffices to estimate $U_{\lambda}(n, t)$. Since $|U_{\lambda}(n, t)| \leq |U_{\lambda}(n, t)| + |U_{\lambda}(n, t) - U_{\lambda}(n, t)|$, we only need to estimate $|U_{\lambda}(n, t)|$ and $|U_{\lambda}(n, t) - U_{\lambda}(n, t)|$, respectively. Since each step involves complicated tools from number theory, harmonic analysis, complex analysis and real analysis, we only list the major results in each steps while omit the detailed proofs.

1) Step 1: Estimate $|U_{\lambda}(n, t)|$. We obtain
\[
\frac{1}{2T} \int_{-T}^{T} |U_{\lambda}(n, t)| dt \leq CC_{r} \varepsilon \lambda \ln^{2} \lambda,
\]
for all $\lambda > 16\pi r^{-4}/b_{\varepsilon}$ and $n \leq \rho_{r} \lambda$.

2) Step 2: Estimate $|U_{\lambda}(n, t) - U_{\lambda}(n, t)|$. We obtain
\[
\frac{1}{2T} \int_{-T}^{T} |U_{\lambda}(n, t) - U_{\lambda}(n, t)| dt \leq CC_{r} r_{\lambda} \lambda \ln \lambda + CC_{r} b_{\varepsilon}^{-1} \lambda.
\]

3) Step 3: Final estimate. Let $\lambda_{e, T} = 16\pi r^{-4}/b_{\varepsilon}$, Combining Steps 1 and 2, by the triangle inequality we obtain that for all $\lambda > \lambda_{e, T}$ and $n \leq \rho_{r} \lambda$,
\[
\frac{1}{2T} \int_{-T}^{T} |U_{\lambda}(n, t)| dt \leq C_{e, \lambda} \lambda \ln \lambda,
\]
where $C_{e, \lambda} := CC_{r} (r_{\lambda} + b_{\varepsilon}^{-1})$. Using the admissible filter, Finally, we have
\[
\frac{1}{2T} \int_{-T}^{T} |e_{\lambda}^*(x, t)| dt \leq C_{e, \lambda} \lambda \lambda^{-3} \varepsilon.
\]
Similarly, the same estimate for $e_{\lambda}^*(x, t)$ can be attained by reflecting the negative indices into the positive indices, which completes the proof of Theorem 1.

IV. CONCLUSION

In this paper, we rigorously analyze the time average mean square error of a sigma-delta system with the inputs of band-limited signals. Combining tools from number theory, harmonic analysis, real analysis and complex analysis, we establish the estimate $O(\lambda^{-3+\varepsilon})$ for any small $\varepsilon > 0$ as the sampling ratio $\lambda$ approaches infinity, which is very close to the long standing conjecture $O(\lambda^{-3})$.

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