Irregular Sampling Theorems for Wavelet Subspaces

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For finite energy γ-band continuous signal $f(t), t \in R$, i.e., $f \in L^2(R)$ and $\text{supp}\hat{f}(\omega) = [-\gamma, \gamma]$, the classical Shannon Sampling Theorem gave the following reconstruction formula,

$$f(t) = \sum_n f(nT) \frac{\sin\gamma(t-nT)}{\gamma(t-nT)}, \qquad T \leq \frac{\pi}{\gamma}, \tag{1}$$

where $\hat{f}(\omega)$ is the Fourier transform of $f(t)$ defined by $\hat{f}(\omega) = \int_R f(t)e^{-i\omega t}dt$. Unfortunately it is not appropriate for non-band-limited signals. However if we let $\gamma = 2^m\pi, m \in Z$, this problem can be viewed as a special case of sampling in wavelet subspaces with $\varphi(t) = \sin\pi t/\pi t$ playing the role of scaling function of MRA $\{V_m = \overline{\text{span}}\{\varphi(2^m t - n)\}_n\}_m$. Realizing these properties, Walter[6] extended (1) to a class of wavelet subspaces. Let $\varphi(t)$ be a continuous scaling function of MRA$\{V_m\}_m$ such that $\varphi(t)| \leq O(|t|^{-1-\varepsilon})$ for some $\varepsilon > 0$ when $|t| \to \infty$. Walter[6] showed that, in orthonormal case, there is a sequence $\{S_k(t)\}_k$ in V_0 such that $S_k(t) = S_0(t-k)$ and

$$f(t) = \sum_k f(k) S_0(t-k) \qquad \text{for } f \in V_0. \tag{2}$$

However the sampling is not always at the same step, or say irregular sampling, then how to deal with it? Paley-Wiener's $\frac{1}{4}$-Theorem (see Young[7]) said that, if $\sup_k |\delta_k| < \frac{1}{4}$ and $\delta_k = -\delta_{-k}$ then for $f(t) \in P_\pi$ (Paley-Wiener Space),

$$f(t) = \sum_k f(k+\delta_k) \frac{G(t)}{G'(k+\delta_k)G(t-(k+\delta_k))}, \tag{3}$$

where $G(t) = t\prod_{n=1}^{\infty}(1 - t^2/(n+\delta_n)^2)$. But it can not well deal with non-band-limited signals, and the sampling with constraints $\delta_k = -\delta_{-k}$ exposes much regularity. Following Walter[6], Liu-Walter[5] tried to extend Paley-Wiener's to the sampling in a class of orthonormal wavelet subspaces without $\delta_k = -\delta_{-k}$. But they could not claim the existence of some $\delta_\varphi \subset (0,1]$ such that a similar reconstruction formula as (3) holds when $\sup_k |\delta_k| < \delta_\varphi$. Then Liu[4] turned to deal with the special case — spline wavelets by Feichtinger-Grochenig Iterative Algorithm (see Feichtinger-Grochenig[2]). Even so, it is to estimate the sampling density, not the deviation. Chen-Itoh-Shiki[1] obtained a recovering formula for general wavelet subspaces without the symmetricity requirement for sampling but lead to an l^1-bound on $\{\delta_k\}_k$, in fact they still can not estimate the above described δ_φ.

In this paper, we provide the reconstruction formulae and establish the algorithm to estimate the deviation bound for irregularly sampled signals in orthogonal and biorthogonal wavelet subspaces respectively after introducing the function class $L_\sigma^\lambda[a,b]$, that does not require the symmetricity constraints $\delta_k = -\delta_{-k}$ of Paley-Wiener's for sampling, but also relaxes its deviation bound in some wavelet subspaces. Then we obtain an irregular sampling theorem and an algorithm for general wavelet subspaces deduced from biorthogonal case. Furthermore the theorems and algorithms are modified to a more useful case by using Zak transform $Z_\varphi(\sigma,\omega)$ (see Janssen[3]). Summarily, we obtain the following main theorem by using the Hilbert reproducing kernel $q_\varphi(s,t) = \sum_k \varphi(s-k)\varphi(t-k)$.

Theorem *Suppose $\{V_m\}_m$ be a Multi Resolution Decomposition of $L^2(R)$ with the continuous scaling function $\varphi(t)$ satisfying*

A. $|\varphi(t)| \leq O(1/|t|^{1+\varepsilon})$ *for some* $\varepsilon > 0$.
B. $Z_\varphi(\sigma,\omega) \neq 0$ *for some* $\sigma \in [0,1)$.
C. $\varphi(t) \in L_\sigma^\lambda[a,b]$ $(\lambda > 0, 0 \in [a,b] \subset [-1,1])$.

Then for any irregular sampling at $\{k+\sigma+\delta_k\}_k$ with $\{\delta_k\}_k \subset [-\delta_{\sigma,\varphi}, \delta_{\sigma,\varphi}] \cap [a,b]$, there exists a $S_{\sigma,k}(t)$ in V_0 such that the original signal $f(t) \in V_0$ can be reconstructed by

$$f(t) = \sum_k f(k+\sigma+\delta_k) S_{\sigma,k}(t), \tag{4}$$

if

$$\delta_{\sigma,\varphi} < \left(\frac{\inf|Z_\varphi(\sigma,\omega)G_\varphi(\omega)| \inf|Z_\varphi(\sigma,\omega)/G_\varphi(\omega)|}{\|q_\varphi(s,\sigma)\|_{L_\sigma^\lambda[a,b]}} \right)^{1/\lambda}, \tag{5}$$

where $G_\varphi(\omega) = \sum_k |\hat{\varphi}(\omega+2k\pi)|^2$.

When the sampling step is not $T=1$, or say $T = 2^{-m}$, the above theorem can be modified to V_m by using the Hilbert reproducing kernel $q_\varphi^{(m)}(s,t) = 2^m \sum_k \varphi(2^m s - k)\varphi(2^m t - k)$.

At the end, we also calculate some famous wavelet subspaces such as B-Spline of order 2, Daubechies' and Meyer's as examples and indicate that δ_φ can be bigger than $\frac{1}{4}$ of Paley-Wiener's for sampling in some wavelet subspaces.

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