AVERAGE ALIASING ERROR FOR GENERAL PREFILTERING BASED ON SHIFT INVARIANT SPACES

Wen Chen

Department of Electronic Engineering
Jiaotong University, Minhang, Shanghai, China
Email: wenchen@sjtu.edu.cn

ABSTRACT
Recently, the conventional A/D conversion has been extended to the A/D conversions based on shift invariant spaces, in which, the prefiltering is performed by quasi-projections into shift invariant spaces and sampling is performed in the shift invariant spaces. This paper studies the accuracy of the extended prefiltering. As a contribution of this paper, a formula to exactly evaluate the average aliasing error is established. Furthermore, for a signal in Sobolev space, removing the restriction that the exponent of Sobolev space is greater than 1/2 imposed by the previous authors, this paper finds that the average aliasing error decays at the rate as the exponent of the Sobolev space.

Index Terms—prefiltering, aliasing error, quasi-projection, shift invariant spaces.

1. INTRODUCTION
In digital signal processing (DSP) and digital communications, an analog signal is converted to a digital signal by an analog-to-digital (A/D) converter. An ideal A/D converter prefilters an analog signal of finite energy by an ideal lowpass filter as illustrated by

\[
\text{Input signal} \rightarrow \text{Ideal lowpass filter} \rightarrow \text{Prefiltered signal},
\]

and produces a discrete time (DT) signal by passing the prefiltered signal through an ideal impulse train. The derived DT signal is then subjected to a quantization process. Since an ideal lowpass filter and an ideal impulse train are impossible to be realized in hardware, a conventional practical A/D converter prefilters an analog signal of finite energy by a non-ideal lowpass filter and passes the prefiltered signal through a non-ideal impulse train.

Recent research on A/D conversion based on shift invariant spaces reveals that the conventional prefiltering by a lowpass filter can be extended and formulated as the prefiltering by quasi-projections into shift invariant spaces [5] as illustrated by

\[
\text{Input signal} \rightarrow \text{Quasi-projection} \rightarrow \text{Prefiltered signal}.
\]

Such an extension of prefiltering is significant and useful because it establishes a theoretical framework for the conventional prefiltering and conversely provides the new methods to design a non-ideal A/D converter of high accuracy and low computational complexity [5].

An important issue in prefiltering is the difference between the prefiltered signal and the original signal, which is referred to as the aliasing error. For an ideal prefiltering, the aliasing error can be easily derived using the Parseval's identity. However, the estimate of aliasing error for the prefiltering by a quasi-projection into shift invariant space involves complicated analysis tools in approximation theory. For a smooth enough signal, certain kind of investigation has been done [3, 2, 10], which claims that the aliasing error decays with respect to the scale of the shift invariant space, at the order of the Strang-Fix condition that the generator satisfies. Since a practical signal is usually non-smooth, the estimate of aliasing error for a non-smooth signal is desirable for theoretical completeness and practical broadness.

The signals considered in [3] are taken from the Sobolev space. Since the only criterion for a signal in Sobolev space is the proper decay of its spectrum, Sobolev space is regarded to be an appropriate signal space for practical signal processing. The exponent of Sobolev space in [3] is assumed to be greater than 1/2, which results in the Hölder continuity of the signal in Sobolev space, and hence excludes the large class of discontinuous signals. In the recent work done in [5], the authors derived an estimate of the aliasing error for the Lipschitz continuous signals in the sense of square norm, which can cover some discontinuous signals. But Lipschitz continuity of a signal in the sense of square norm is inconvenient to verify in practical manipulation.

For the practical purpose of convenience and the theoretical purpose of generality, it is desirable to establish the estimate of aliasing error for the signals in Sobolev space, but remove the restriction on the exponent of the Sobolev space imposed in [3]. One of our objectives in this paper is to establish such an estimate. Removing the restriction on the exponent of Sobolev space, this paper derives a formula to exactly evaluate the average aliasing error. Furthermore, for a signal in Sobolev space, the average aliasing error is found to decay at the order as the exponent of the Sobolev space. In addition, we have derived the coefficient of the decay rate, which has lower computational complexity than the previous ones [3, 5].

2. PREFILTERING BY QUASI-PROJECTIONS INTO SHIFT INVARIANT SPACES
In this section, we will set up the prefiltering by quasi-projections into shift invariant spaces and introduces the related symbols and concepts, such as shift invariant space, Sobolev space, aliasing error, Fourier transform (FT), signals of finite energy, bandlimited signals and the Dirac signal.
2.1. Signals of finite energy, Fourier transform, bandlimited signals and the Dirac signal

An analog signal \( f \) is of finite energy if the square norm \( \| f \|_2 = \left( \int_{\mathbb{R}} |f(t)|^2 \, dt \right)^{1/2} < \infty \). We also denote by \( L^2(\mathbb{R}) \) the signal space of finite energy, that is, \( L^2(\mathbb{R}) = \{ f : \| f \|_2 < \infty \} \). \( f \) is said to be bandlimited if \( \mathcal{F}(\omega) = 0 \) whenever \( |\omega| > \sigma \) for some \( \sigma > 0 \), where \( \mathcal{F} \) is the spectrum or Fourier transform (FT) of \( f \) defined by \( \mathcal{F}(\omega) = \int_{\mathbb{R}} f(t) e^{-j2\pi \omega t} \, dt \). In this case, \( f \) is called a \( \sigma \)-band signal. The continuous time (CT) Dirac signal \( \delta \) is defined as \( \delta(t) = 0 \) for \( t \neq 0 \) and \( \int_{\mathbb{R}} \delta(t) \, dt = 1 \); The discrete time (DT) Dirac signal is defined as \( \delta[n] = 1 \) and \( \delta[n] = 0 \) for \( n \neq 0 \).

2.2. Shift invariant spaces

For \( \alpha \geq 1 \), the (scaled) shift invariant space (SIS) \( \mathcal{V}_\alpha(\varphi) \) generated by the generator \( \varphi \in L^2(\mathbb{R}) \) is defined as \([2,9]\),

\[
\mathcal{V}_\alpha(\varphi) := \left\{ \sum \epsilon^j \varphi(\lambda - \ell) : \sum |\epsilon^j|^2 < \infty \right\} \subset L^2(\mathbb{R}),
\]

where \( \alpha \), called the scale of the SIS \( \mathcal{V}_\alpha(\varphi) \), is understood to be bandwidth in prefiltering or sampling ratio in sampling. In this paper, we also assume that \( \{\varphi(\lambda - \ell)\} \) is a Riesz basis of \( \mathcal{V}_\alpha(\varphi) \), that is, \( \alpha < G_\varphi < b \) almost everywhere for some positive constants \( a \) and \( b \) [2], where \( G_\varphi := \sum \|\varphi(\cdot + \ell)\|^2 \) is called the ratio of orthornormality for the Riesz basis \( \{\varphi(\cdot - \ell)\} \). Taking \( \text{ sinc } t = \frac{\sin \pi t}{\pi t} \), the SIS \( \mathcal{V}_\lambda(\text{sinc}) \) is exactly the \( \lambda \)-band signal space of finite energy.

2.3. Prefiltering and aliasing error

The conventional ideal prefiltering by an ideal lowpass filter boils down to a quasi-projection \( P^\text{sinc}_\varphi : L^2(\mathbb{R}) \to \mathcal{V}_\lambda(\text{sinc}) \), defined by,

\[
P^\text{sinc}_\varphi(f) := \lambda \sum \epsilon^j \langle f, \text{sinc}(\lambda - \ell) \rangle \text{sinc}(\lambda - \ell), \quad \forall f \in L^2(\mathbb{R}),
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^2(\mathbb{R}) \) defined by \( \langle f, g \rangle = \int_{\mathbb{R}} f(t) g^*(t) \, dt \). Hence the aliasing error is \( \epsilon^j = f - P^\text{sinc}_\varphi(f) \), which can be made arbitrarily small by increasing the scale \( \lambda \) of the SIS \( \mathcal{V}_\lambda(\text{sinc}) \), i.e., the bandwidth of the lowpass filter (see equation (1)).

Observation is very essential to establishing the sampling theory in shift invariant spaces [4, 6, 7, 8, 12, 13], and the prefecttering theory based on shift invariant spaces [3, 5], that will be also addressed in this paper. By replacing the generator \( \text{sinc} \) by a general generator \( \varphi \), one can consider prefiltering a signal by a quasi-projections into a SIS (see equation (2)), that is, to project a signal of finite energy to a SIS \( \mathcal{V}_\lambda(\varphi) \) by a quasi-projection \( P^\varphi_\lambda(f) \) in \( L^2(\mathbb{R}) \) defined by

\[
P^\varphi_\lambda(f) := \lambda \sum \epsilon^j \langle f, \varphi(\lambda - \ell) \rangle \varphi(\lambda - \ell), \quad \forall f \in L^2(\mathbb{R}).
\]

Then the aliasing error is defined as \( \epsilon^j = f - P^\varphi_\lambda(f) \). For the simplicity of realization, we do not consider the dual or integral operator version of quasi-projection in this paper [3, 10].

Taking the small random delay of the initial input to a practical prefiltering system or the small random initial phase of a practical sampler into account, it is reasonable to measure some kind of average aliasing error over a small delay interval or a small sampling period. Assume that the random delay of the initial input or the random initial phase of the sampler is uniformly distributed. Then the average aliasing error \( \bar{\epsilon}^j \) is defined as

\[
\bar{\epsilon}^j := \left\{ \lambda \int_{\mathbb{R}} \| \epsilon^j_\varphi \|_2^2 \, du \right\}^{1/2},
\]

where the delayed signal \( f_\varphi \) is defined as \( f_\varphi = f(\cdot - \varphi) \).

2.4. Sobolev spaces

Let \( \lambda \) be a positive real number. The \textit{Sobolev space} \( W^\lambda \) consists of all measurable functions \( f \) satisfying \( \int_{\mathbb{R}} |\varphi|^2 \langle f(\omega) \rangle^2 \, d\omega < \infty \). In line with the definition of regularity, we use the notation \( \| f \|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\varphi|^2 \langle f(\omega) \rangle^2 \, d\omega \). Sobolev space is a spread and appropriate signal space for the practical signals since the spectrum of signal in a Sobolev space decays appropriately. For example, in the conventional practical sampling where the actual sampling rate is less than the Nyquist sampling rate, it is appropriate to take the signal into a Sobolev space if the spectrum of the signal decays appropriately.

2.5. Strang-Fix Condition

In this paper, we also consider the \textit{Strang-Fix condition} which has been widely used in approximation by shift invariant spaces [2, 10, 11]. A generator \( \varphi \) is said to satisfy the \( L \)-th order \textit{Strang-Fix condition} if its spectrum \( \tilde{\Phi} \) satisfies \( \tilde{\Phi}(0) \neq 0 \) and \( \tilde{\Phi}(\ell) = 0 \) for all integer \( \ell \neq 0 \) and \( \alpha = 0, \ldots, L - 1 \). We also assume that \( \sum_i |\ell|^j |\varphi(\ell + t)| < \infty \), so that \( \tilde{\Phi} \) has the bounded derivatives up to \( L \).

3. AVERAGE ALIASING ERROR FOR PREFILTERING BY QUASI-PROJECTIONS INTO SHIFT INvariant SPACES

In this section, we are going to evaluate the average aliasing error for prefiltering by a quasi-projection into a shift invariant space. Furthermore, we will use the derived formula to estimate the average aliasing error for a signal in Sobolev space. Finally a numerical example based on the symmetric B-splines is calculated to demonstrate the good performance of general prefiltering as compared to the conventional prefiltering.

3.1. Average aliasing error

In this subsection, we are going to evaluate the average aliasing error for a signal of finite energy prefiltered by a quasi-projection into shift invariant space. For a generator \( \varphi \), we define the kernel function \( K_\varphi \) using the spectrum \( \tilde{\Phi} \) of \( \varphi \) as \( K_\varphi := 1 - 2|\tilde{\Phi}|^2 + |\tilde{\Phi}|^4 G_\varphi \). It is of interest to note that the kernel can be simplified to \( K_\varphi = 1 - |\tilde{\Phi}|^2 \) if \( \varphi \) is orthonormal. In particular, \( K^\text{sinc} = 1 - \chi[{-1/2,1/2}] \) for the ideal prefiltering. Using the kernel \( K_\varphi \) and combining the tools from approximation theory and classical analysis, we obtain the following simple formula to exactly calculate the average aliasing error.

\[
\text{Theorem 1 If a signal } f \text{ of finite energy is prefiltered by the quasi-projection } P^\varphi_\lambda \text{ into the SIS } \mathcal{V}_\lambda(\varphi), \text{ then the average aliasing error is } \epsilon^j = \left\{ \int_{\mathbb{R}} \| f(\omega) \|^2 K_\varphi \langle \varphi \rangle^2 \, d\omega \right\}^{1/2}, \text{ where } F \text{ is the spectrum of } f.
\]
Compared to what obtained in [3], Theorem 1 removes the restrictions \( f \in W^r \) and \( r > 1/2 \). Since a practical signal is usually not smooth or continuous, this result, because of its applicability to any signal of finite energy, establishes a more general framework for the extended prefiltering. Theorem 1 also indicates that one can exactly calculate the average aliasing error for the prefiltering by quasi-projections. For example, the average aliasing error \( \tilde{e}_r \) is given by 
\[
\tilde{e}_r = \left\{ \int_0^\infty \left[ |F(\omega)|^2 + |F(-\omega)|^2 \right] d\omega \right\}^{1/2}
\]
for the ideal prefiltering (\( \varphi = \text{sinc} \)), which coincides with the conventional argument on this issue, and hence can be made arbitrarily small by increasing the bandwidth \( \lambda \).

3.2. Signals in Sobolev spaces

For the ideal prefiltering, by the argument in the last sub-section, if the spectrum \( F \) decays at some rate, say \( f \in W^r \) for some \( r > 0 \), then the average aliasing error satisfies
\[
\tilde{e}_r \leq \lambda^{-r} \left\{ \int_0^\infty \left[ |F(\omega)|^2 + |F(-\omega)|^2 \right] d\omega \right\}^{1/2} \leq \lambda^{-r} \left\| f \right\|_2.
\]
In the same spirit, it is desirable to establish a similar estimate of the average aliasing error for the prefiltering by quasi-projections into shift invariant spaces if \( f \) belongs to the Sobolev space \( W^r \) for some \( r > 0 \). This is expressed in the following theorem.

**Theorem 2** Assume that the kernel satisfies \( \left\| K_\varphi \right\|_{-2r} < \infty \). Then the average aliasing error satisfies
\[
\tilde{e}_r \leq \lambda^{-r} \left\| f \right\|_2 \left\| K_\varphi \right\|_{-2r} < \infty \text{ if a signal } f \text{ in the Sobolev space } W^r \text{ for some } r > 0 \text{ is prefiltered by the quasi-projection } P_\varphi^r.
\]

Therefore the average aliasing error decays at the rate as the exponent of the Sobolev space. Compared to the error estimate for ideal prefiltering, there are additional multiplier \( \left\| K_\varphi \right\|_{-2r} \) in the average aliasing error for the general prefiltering. For \( \varphi = \text{sinc} \), the additional multiplier term \( \left\| K_\text{sinc} \right\|_{-2r} \) is given by
\[
\left\| K_\text{sinc} \right\|_{-2r} = \left\| \text{d(1-\chi(-\lambda))} \cdot \text{sine} \right\|_{-2r} = 1,
\]
which shows that this estimate covers the conventional ideal prefiltering.

The next step is to find the conditions on the generator \( \varphi \) such that \( \left\| K_\varphi \right\|_{-2r} < \infty \). It is worth of indicating that a necessary condition for \( \left\| K_\varphi \right\|_{-2r} \) is to make \( K_0(0) = 0 \), i.e., \( G_0(0) = 0 \). Technically, it is easy to choose a generator such that \( \Phi(0) = 0 \) by normalizing it as \( \Phi(\lambda) = \Phi(0) \). Without loss of generality, we can assume that \( \Phi(0) = 1 \) for a general generator. Combining with the necessary condition \( K_0(0) = 0 \), we have \( G_0(0) = 1 \). Then \( \Phi(\lambda) = \Phi(0) \) for any \( \lambda \), which exactly implies that \( \varphi \) satisfies the first order Strang-Fix condition. Conversely \( \Phi(\lambda) = \Phi(0) \) if \( \varphi \) satisfies the first order Strang-Fix condition. Then \( K_\varphi = \Omega(\varphi^2) \) by writing \( K_\varphi = \left\{ 1 - |\Phi|^2 \right\}^2 + |\Phi|^2 \left\{ \delta(\Phi + 0) + \delta(\Phi - 0) \right\} \), and hence \( \left\| K_\varphi \right\|_{-2r} < \infty \) for \( r > 0 \). This is shown the following Corollary.

**Corollary 1** Suppose that \( \Phi(0) = 1 \). Then the generator \( \varphi \) satisfies the first order Strang-Fix condition if and only if \( \left\| K_\varphi \right\|_{-2r} < \infty \) for some positive number \( r < 1 \). Consequently, the average aliasing error satisfies \( \tilde{e}_r \leq \lambda^{-r} \left\| f \right\|_2 \left\| K_\varphi \right\|_{-2r} < \infty \text{ if a signal } f \text{ in the Sobolev space } W^r \text{ is prefiltered by the quasi-projection } P_\varphi^r.
\]

In general, if \( \varphi \) satisfies the \( L \)th order Strang-Fix condition for some positive integer \( L \), we have the following criterion.

**Corollary 2** Assume that the generator \( \varphi \) satisfies the \( L \)th order Strang-Fix condition. Then there is a \( \psi \), a finite linear combination of the shifts of \( \varphi \), such that \( \left\| K_\psi \right\|_{-2r} < \infty \). Consequently, the average aliasing error satisfies \( \tilde{e}_r \leq \lambda^{-r} \left\| f \right\|_2 \left\| K_\psi \right\|_{-2r} < \infty \text{ if a signal } f \text{ in the Sobolev space } W^r \text{ for some } r > 0 \text{ is prefiltered by the quasi-projection } P_\psi^r \text{ into } V_\psi(\varphi) \text{ and } r < L.
\]

Intuitively it seems that Strang-Fix condition is too strong for obtaining the decay rate of the average aliasing error. But in some sense, the Strang-Fix condition is a necessary condition for the average aliasing error to decay at some rate [3, 2, 10].

3.3. Numerical results

We give a numerical example to demonstrate the prefiltering based on the symmetric B-spline shift invariant space. A symmetric B-spline \( \beta^N \) of degree \( N \) is defined as the \( N \)-times convolution of the characteristic function \( \chi_{-1/2,1/2}[x] \), i.e., \( \beta^N := \chi_{-1/2,1/2} * \cdots * \chi_{-1/2,1/2} \). Then Fourier transform of \( \beta^N \) is \( \text{sinc}^N \). Since \( K_{\beta^N} = 1 - 2 \text{sinc}^{2N} + \text{sinc}^{2N} G_{\beta^N}, \) by Theorem 1, the average aliasing error is
\[
\tilde{e}_r = \int_{-\pi}^{\pi} |F(\omega)| \left[ 1 - 2 \text{sinc}^{2N} \left( \frac{\omega}{\lambda} \right) + \text{sinc}^{2N} \left( \frac{\omega}{\lambda} \right) G_{\beta^N} \left( \frac{\omega}{\lambda} \right) \right] d\omega,
\]
if a signal \( f \) of finite energy is prefiltered by the quasi-projection \( P_{\beta^N}^r \) into the SIS \( V_{\beta^N}(\beta^N) \).

Suppose that a signal \( f \) is taken from the Sobolev space \( W^r \) for some \( r > 0 \), and prefiltered by the quasi-projection \( P_{\beta^N}^r \). Since Matlab shows that \( \left\| K_{\beta^N} \right\|_{-4r} \leq 11N^2, \) by Corollary 2, the average aliasing error satisfies \( \tilde{e}_r \leq 11N^2 \lambda^{-r} \left\| f \right\|_2 \) if \( r < 2 \). Noting that the \( \beta^N \) satisfies the \( N \)th order Strang-Fix condition, by Corollary 2, there is a \( \psi^N \), a finite linear combination of the shifts of \( \beta^N \), such that the average aliasing error satisfies \( \tilde{e}_r \leq \lambda^{-r} \left\| f \right\|_2 \left\| K_{\psi^N} \right\|_{-2r} < \infty \), if \( f \) is prefiltered by \( P_{\beta^N}^r \) and \( r \leq 2 \).
N. In the following, we try to figure out the $\psi^N$ for the small $N$, e.g., $N = 2, 3$. Since $(\sin \psi)^{\psi}(0) = 0$, we can take $\psi^2 = \beta^2$. Since $(\sin \psi)^{\psi}(0) = 0$ and $(\sin \psi)^{\psi} = -1$, using $\Psi^{\psi}(0) = \sum_{k=0}^{\infty} \frac{1}{k!} \beta^{k+1} f^{(k)}(0) \beta^{k+1} = \delta(\alpha)$ for $\alpha = 0, 1, 2, \ldots$, we have $P_2(0) = 1$. $P_2(0) = 0, P_2(0) = 1$. This solves $\psi = \frac{\psi}{\psi + 1}$ and $\psi = \frac{1}{\psi + 1}$. Therefore $\psi = \frac{\psi}{\psi + 1}$ and $\psi = \frac{1}{\psi + 1}$. We leave the interesting calculation for $N > 3$ to the interested readers. We now turn to consider prefilters in the physical signal based on the symmetric B-spline of degree 2 and compare it with the ideal prefilters.

Consider the input signal $f$ as shown in Fig. 1 (a). The spectrum $F$ of $f$ is shown in Fig. 1 (b). Empirical estimate shows that $F(\omega) = O(\omega^{-2})$ at infinity (see Fig. 1 (b)), and hence $f \in W^r$ for $r < 2$. For simplicity, we use the prefilters $P_2(\beta)$ (see Fig. 1 (c)) for the graph of $\beta^2$. Since Matlab shows that $K_{\beta^2}(\omega)/|\omega|^4 \leq 45\beta^2$, the average aliasing error satisfies $\|f^2\|_{L^2}^\alpha \leq 45\beta^2$, for $r < 2$. Fig. 1 (d) shows this estimate versus the actual aliasing error for $r = 3, \ldots, 20$. The actual aliasing error is seen to be lower than the estimate, which implies that our estimate is not yet optimal. We also put the actual aliasing error for the ideal lowpass prefilttering in Fig. 1 (d) for comparison. It is observed that the actual accuracy of prefilttering by $P_2(\beta)$ is superior to that by $P_2(\beta)$, even if their theoretical estimates are in the same order. Visually, we see that the prefilttering by a quasi-projector into $V^\alpha(\beta^2)$ provides good approximation.

Fig. 2, the ideal lowpass prefilttering $P_2(\beta^2)$ introduces ripples in the smooth part of $f$. However, $P_2(\beta^2)$ approximates $f$ very well. Since $\beta^2$ is compactly supported, this means reduction of the computational complexity in prefilttering.