Supprrium of Perturbation for Sampling in Shift Invariant Subspaces

Wen Chen*, Shuichi Itoh, Junji Shiki

Dept. of Information Network Sciences, University of Electro-Communications
Chofugaoka 1-5-1, Chofu 182-8585, Tokyo, Japan

ABSTRACT

In the more general framework ``shift invariant subspace", the paper obtains a different estimate of sampling in function subspace to our former work, by using the Frame Theory. The derived formula is easy to be calculated, and the estimate is relaxed in some shift invariant subspaces.

Keywords: Sampling, Shift Invariant Subspace, Generating Function, Frame, Zak-transform

1. INTRODUCTION

In digital signal and image processing, digital communications, etc., a continuous signal is usually represented and processed by using its discrete samples. Then a fundamental question is how to represent a signal in terms of a discrete sequence. The famous classical Shannon Sampling Theorem describes that a finite energy band-limited signal is completely characterized by their sample values. Realizing that the Shannon function $\sin c(t) = \sin(t)/t$ is in fact a scaling function of an MRA (Multi-resolution Analysis), Walter[18] found a sampling theorem for a class of wavelet subspaces. Following Walter[18]'s work, Janssen[13] studied the shift sampling in Wavelet subspaces by using Zak-transform. Xia-Zhang[22] discussed the so-called sampling property. Walter[19], Xia[21] and Chen-Itoh[7,8] studied the more general case ``over-sampling." On the other hand Aldroubi-Unser[1, 2,3] and Unser-Aldroubi[17] studied the sampling procedure in shift-invariant subspaces. Chen-Itoh[9] improved Walter[18] and Aldroubi-Unser[3]'s works, and we found a general sampling theorem for shift-invariant subspace.

However, in many real applications samplings are not always made regularly. Sometimes the sampling steps need to be fluctuated according to the signals so as to reduce the number of samples and the computation complexity. There are also many cases where undesirable jitter exists in sampling instants. Some communication systems may suffer from the random delay due to the channel traffic congestion and encoding delay. In such cases, the system will be made to be more efficient if the irregular factor is considered. Then how are these irregularly sampled signals could be dealt with? For the finite energy band-limited signals, a generalization of Shannon Sampling Theorem, known as the Codec Theorem[23], can be used. Following the works on sampling in wavelet subspace, Liu-Walter[15], Liu[14], and Chen-Itoh-Shiki[4] extended Codec Theorem to a class of wavelet subspaces. But their results are not mild. Then Chen-Itoh-Shiki[6] introduced a function class $L_\sigma^\lambda[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$, and $0 \in [a,b] \subset [-1,1]$) and a norm $\|\bullet\|_{L_\sigma^\lambda[a,b]}$ of $L_\sigma^\lambda[a,b]$. Finally we found an irregular sampling theorem for wavelet subspaces with an $L_\sigma^\lambda[a,b]$-scaling function as the following.

*Correspondence: Email: wchen@net.is.uec.ac.jp; Tel: 0424-42-8091; Fax: 0424-42-8092

Wavelet Applications in Signal and Image Processing VIII, Akram Aldroubi, Andrew F. Laine,
Michael A. Unser, Editors, Proceedings of SPIE Vol. 4119 (2000) © 2000 SPIE · 0277-786X/00/$15.00

Theorem [6]
Suppose a continuous $L_\sigma^\lambda[a,b]$-scaling function $\varphi(t)$ of an MRA $\{V_m\}_m$ is such that

$$Z_\varphi(\sigma,\omega) \neq 0,$$
$$\varphi(t) = O(|t|^{-s}), \text{ for some } s > 1.$$

Then there is a $\delta_{\sigma,\varphi} \in (0,1]$ such that for any sequence $\{\delta_k\}_k \subset [-\delta_{\sigma,\varphi}, \delta_{\sigma,\varphi}] \cap [a,b]$, there is an sequence $\{S_{\sigma,k}(t)\}_k \subset V_0$ such that

$$f(t) = \sum_k f(k+\sigma+\delta_k) S_{\sigma,k}(t)$$

holds if

$$\delta_{\sigma,\varphi} < \left(\left\| Z_\varphi(\sigma,\omega) G_\varphi(\omega) \right\|_0 \left\| Z_\varphi(\sigma,\omega) / G_\varphi(\omega) \right\|_0 \Big/ \left\| q_{\varphi(s,\sigma)} \right\|_{L_\sigma^\lambda[a,b]} \right)^{1/\lambda}.$$

Applying the theorem to calculating the B-spline of order 1 scaling function $N_1(t) = t\chi_{[0,1)} + (2-t)\chi[1,2)$, we find $\delta_{0,N_1} < \sqrt{3}/2$ when all $\delta_k > 0$ or all $\delta_k < 0$. But Liu-Walter[15] found that the superrium of perturbation for B-spline of order 1 could be 1/2, i.e., $\delta_{0,N_1} < 1/2$, and they also showed that 1/2 is the optimal superrium of perturbation for sampling in space generated by $N_1(t)$. This implies that Chen-Itoh-Shiki[6]'s result is not at least optimal.

Our purpose in this paper is trying to find the optimal $\delta_{\sigma,\varphi}$ such that the aforementioned reconstruction formula holds. We would like to consider the sampling in the more general framework ``shift invariant subspaces." In this framework we obtain a different estimate of $\delta_{\sigma,\varphi}$ by using Frame Theory. By applying the new result to calculating the B-spline of order 1, we find $\delta_{0,N_1} < 1/2$ when all $\delta_k > 0$ or all $\delta_k < 0$.

Let us now roughly introduce the shift invariant subspaces and the frame Theory respectively. For $\varphi(t) \in L^2(R)$, let

$$V(\varphi) = \{\textstyle\sum_k c_k \varphi(t-k) : \{c_k\}_k \in l^2\}.$$

In general $\{\varphi(t-k)\}_k$ is not a Riesz basis of $V_0(\varphi)$. In fact $\{\varphi(t-k)\}_k$ is a Riesz basis of $V_0(\varphi)$ if and only if

$$0 < \left\| G_\varphi(\omega) \right\|_0 \leq \left\| G_\varphi(\omega) \right\|_\infty < \infty,$$

where

$$G_\varphi(\omega) = \left(\textstyle\sum_k |\hat{\varphi}(\omega + 2k\omega)|^2 \right)^{1/2},$$

and $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(t)$ defined by

$$\hat{\varphi}(\omega) = \int_R \varphi(t) e^{-it\omega}\, dt.$$

If $\{\varphi(t-k)\}_k$ is a Riesz basis of $V_0(\varphi)$, $\varphi(t)$ is called a *generating function,* and $V_0(\varphi)$ called a *shift invariant subspace*. The $\{\varphi(t-k)\}_k$ is an orthonormal basis of $V_0(\varphi)$ if and only if $G_\varphi(\omega) = 1$ $(a.e.)$. In this case $\varphi(t)$ is called an *orthonormal generating function* and $V_0(\varphi)$ called an *orthonormal shift invariant subspace*[12, 3].

A function sequence $\{S_n(t)\}_n$ of a subspace H of $L^2(R)$ is called a frame of H if there is a constant $C \geq 1$ such that

$$C^{-1}\|f\|^2 \leq \sum_n \left| \langle f(t), S_n(t) \rangle \right|^2 \leq C\|f\|^2$$

holds for any $f(t) \in H$. Obviously a Riesz basis is a frame. Moreover there must exist an unique frame $\{\widetilde{S}_n(t)\}_n$ of H (called the *dual frame* of $\{S_n(t)\}_n$) such that

$$f(t)=\sum_n \langle f(t),S_n(t)\rangle \widetilde{S}_n(t)=\sum_n \langle f(t),\widetilde{S}_n(t)\rangle S_n(t)$$

always holds for any $f(t)\in H$ [23].

The following are some notations used in this paper. For a measurable set E, $|E|$ denotes the measure of E. For the measurable functions $f(t)$ and $g(t)$, we write

$$\langle f(t),g(t)\rangle=\int_R f(t)g(t)dt,$$

$$\|f\|=\sqrt{\langle f(t),g(t)\rangle},$$

$$\|f\|_\pi=\sqrt{\int_0^{2\pi}|f(t)|^2},$$

$$\|f\|_0=\sup_{|E|=0}\inf_{R\setminus E}|f(t)|,$$

$$\|f\|_\infty=\inf_{|E|=0}\sup_{R\setminus E}|f(t)|,$$

$$q_f(s,t)=\sum_n f(s-n)f(t-n),$$

$$\hat{f}^*(\omega)=\sum_n f(n)e^{-in\omega}.$$

2. A SAMPLING THEOREM FOR SHIFT INVARIANT SUBSPACES

When we want to find a method to reconstruct a signal $f(t)$ by using their samples $\{f(t_k)\}_k$, obviously the samples can not be arbitrary, i.e., some constraints should impose on $\{f(t_k)\}_k$. The weaker the constraints are the better the reconstruction method will be. Our purpose in this section is to find some weak constraints for reconstructing signals from their discrete samples. Fortunately we found a near necessary-sufficient condition such that a reconstruction formula like the one in theorem of section 1 holds. This result will be also applied to the following sections on the perturbation of sampling in the integer points.

Theorem
Suppose a generating function $\varphi(t)$ of a shift invariant subspace $V_0(\varphi)$ is such that $\{\varphi(t_n-k)\}_k\in l^2$ for any integer n ($n\in Z$). Then there is a frame $\{S_n(t)\}_n$ of $V_0(\varphi)$ such that

$$f(t)=\sum_n f(t_n)S_n(t)$$

holds for any $f(t)\in V_0(\varphi)$ if there is a constant $C\geq 1$ such that

$$C^{-1}\|f\|^2\leq\sum_n|f(t_n)|^2\leq C\|f\|^2$$

holds for any $f(t)\in V_0(\varphi)$.

Proof
Take $g(t)$ such that

$$\hat{g}(\omega)=\hat{\varphi}(\omega)G_\varphi^{-1}(\omega).$$

Then $g(t)$ is an orthonormal generating function of the shift invariant subspace $V_0(\varphi)$ [12]. Suppose

$$G_\varphi^{-1}(\omega)=\sum_k g_k e^{-ik\omega}.$$

Then

$$g(t)=\sum_k g_k \varphi(t-k),$$

and

$$\begin{aligned}
&\left(\sum_k |g(t_n-k)|^2\right)^{1/2} \\
&=\frac{1}{2\pi}\left\|\sum_k g(t_n-k)e^{ik\omega}\right\|_\pi \\
&=\frac{1}{2\pi}\left\|\sum_k \sum_l g_l \varphi(t_n-k-l)e^{ik\omega}\right\|_\pi \\
&=\frac{1}{2\pi}\left\|\sum_l g_l e^{-ik\omega}\sum_k \varphi(t_n-k-l)e^{i(k+l)\omega}\right\|_\pi \\
&=\frac{1}{2\pi}\left\|G_\varphi^{-1}(\omega)\sum_k \varphi(t_n-k)e^{ik\omega}\right\|_\pi \\
&\le \frac{1}{2\pi}\left\|G_\varphi^{-1}(\omega)\right\|_\infty \left\|\sum_k \varphi(t_n-k)e^{ik\omega}\right\|_\pi \\
&=\left\|G_\varphi^{-1}(\omega)\right\|_\infty \left(\sum_k |\varphi(t_n-k)|^2\right)^{1/2}
\end{aligned}$$

Therefore $\{g(t_n-k)\}_k \in l^2$ due to $\{\varphi(t_n-k)\}_k \in l^2$. Let

$$q_g(t,t_n)=\sum_k g(t-k)g(t_n-k)$$

Then $q_g(t,t_n)$ is well defined and $q_g(t,t_n)\in V_0(\varphi)$ due to $\{g(t-n)\}_n$ is a Riesz basis of $V_0(\varphi)$. For any $f(t)\in V_0(\varphi)$, there must be a $\{c_k\}_k \in l^2$ such that $f(t)=\sum_k c_n g(t-k)$. Following the Parseval Identity, we derive

$$\begin{aligned}
&\left\langle f(t), q_g(t,t_n)\right\rangle \\
&=\frac{1}{2\pi}\int_R \hat{f}(\omega)\overline{\sum_k g(\omega)g(t_n-k)e^{ik\omega}}\,d\omega \\
&=\frac{1}{2\pi}\int_R \hat{g}(\omega)\sum_k c_k e^{-ik\omega}\overline{\sum_k \hat{g}(\omega)g(t_n-k)e^{-ik\omega}}\,d\omega \\
&=\frac{1}{2\pi}\int_R \hat{g}(\omega)\sum_k c_k e^{-ik\omega}\overline{\sum_k \hat{g}(\omega)g(t_n-k)e^{-ik\omega}}\,d\omega \\
&=\frac{1}{2\pi}\int_{[0,2\pi]} G_g^2(\omega)\sum_k c_k e^{-ik\omega}\overline{\sum_k g(t_n-k)e^{-ik\omega}}\,d\omega \\
&\le \frac{1}{2\pi}\int_{[0,2\pi]} \sum_k c_k e^{-ik\omega}\overline{\sum_k g(t_n-k)e^{-ik\omega}}\,d\omega \\
&=\sum_k c_k g(t_n-k) \\
&=f(t_n)
\end{aligned}$$

Here we used the fact $G_g(\omega)=1$ $(a.e.)$. Hence

$$C^{-1}\|f\|^2 \le \sum_n \left\langle f(t), q_g(t,t_n)\right\rangle \le C\|f\|^2$$

holds for any $f(t)\in V_0(\varphi)$. It means that $\{q_g(t,t_n)\}_n$ is a frame of $V_0(\varphi)$. Thus there is a dual frame $\{S_n(t)\}_n$ of $\{q_g(t,t_n)\}_n$ in $V_0(\varphi)$. such that

$$f(t)=\sum_n \left\langle f(t), q_g(t,t_n)\right\rangle S_n(t)=\sum_n f(t_n)S_n(t)$$

holds for any. $f(t)\in V_0(\varphi)$

Remark:

1. On the contrary if $\{S_n(t)\}_n$ in $V_0(\varphi)$ is the frame such that the construction formula in this theorem holds, there is also a dual frame $\{\hat{S}_n(t)\}_n$ of $V_0(\varphi)$ such that

$$C^{-1}\|f\|^2 \le \sum_n \left|\left\langle f(t), \hat{S}_n(t)\right\rangle\right|^2 \le C\|f\|^2$$

holds for some $C \ge 1$ and any $f(t) \in V_0(\varphi)$. In general, the dual frame is not biorthogonal to the frame if it is not a independent frame. Now we assume that $\{S_n(t)\}_n$ is independent, then

$$\begin{aligned}
&\left\langle \tilde{S}_n(t), f(t)\right\rangle \\
&= \left\langle \tilde{S}_n(t), \sum_k f(t_k) S_k(t)\right\rangle \\
&= \sum_k f(t_k)\left\langle \tilde{S}_n(t), S_k(t)\right\rangle \\
&= f(t_n)
\end{aligned}$$

This implies that our condition is also necessary. So we call it being near sufficient-necessary.

2. The $\{S_n(t)\}_n$ is the solution of the equations

$$\left\langle S_m(t), \hat{\varphi}(\omega) G_\varphi^{-2}(\omega) \sum_k \varphi(t_n - k) e^{ik\omega}\right\rangle = 2\pi\delta_{n,m}$$

This is because that $\{S_n(t)\}_n$ is biorthogonal to $\{q_g(t,t_n)\}_n$ and

$$\begin{aligned}
&\hat{q}_g(t_n, \omega) \\
&= \sum_k g(t_n - k)\hat{g}(\omega) e^{-ik\omega} \\
&= \sum_k \sum_l c_l \varphi(t_n - k - l) G_\varphi^{-1}(\omega)\hat{\varphi}(\omega) e^{-ik\omega} \\
&= G_\varphi^{-1}(\omega)\hat{\varphi}(\omega) \sum_k \sum_l c_l e^{il\omega} \varphi(t_n - k - l) e^{-i(k+l)\omega} \\
&= G_\varphi^{-1}(\omega)\hat{\varphi}(\omega) \sum_l c_l e^{il\omega} \sum_k \varphi(t_n - k) e^{-ik\omega} \\
&= G_\varphi^{-2}(\omega)\hat{\varphi}(\omega) \sum_k \varphi(t_n - k) e^{-ik\omega}
\end{aligned}$$

3. IRREGULAR SAMPLING IN SHIFT INVARIANT SUBSPACES

An important case of sampling is the perturbation of the regular sampling, i.e., $t_n = n + \delta_n$ ($\delta_n \in (-1,1)$). A fundamental question in this case is how to estimate the superrium of the perturbation $\{\delta_n\}_n$. Following Codec Theorem for finite energy band-limited signals, we have given an estimate for wavelet subspace by using the Riesz basis theory in our former works[6]. In the following, we obtain a different estimate by using the Frame Theory, which is demonstrated by an example to be relaxed in some sense.

In order to establish the theorem, we also need to introduce the function class $L_\sigma^\lambda[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$, and $0 \in [a,b] \subset [-1,1]$) given and used in our former work[6]. We have reasoned that the class is a proper collection by giving some propositions in that paper. Here we only repeat the definition.

Definition

A function $f(t)$ is an element of $L_\sigma^\lambda[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$, and $0 \in [a,b] \subset [-1,1]$) if there is a constant $C_{\sigma,f} > 0$ such that for any $\delta_k \in [a,b]$,

$$\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)| \le C_{\sigma,f} \sup_k |\delta_k|^\lambda .$$

In this case, we also write

$$\|\bullet\|_{L_\sigma^\lambda[a,b]} = \sup_k \frac{\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)|}{\sup_k |\delta_k|^\lambda} .$$

Theorem

Suppose a generating function $\varphi(t)$ of a shift invariant subspace $V_0(\varphi)$ is such that

1. $C^{-1} \le |\hat{\varphi}^*(\omega)| \le C$ $(a.e.)$, for some constant $C \ge 1$,
2. $\varphi(t) \in L_0^\lambda[a,b]$.

Then for any $\{\delta_k\}_k \subset [-\delta_\varphi, \delta_\varphi] \cap [a,b]$, there is a frame $\{S_n(t)\}_n$ of $V_0(\varphi)$ such that

$$f(t) = \sum_n f(n+\delta_n) S_n(t)$$

holds for any $f(t) \in V_0(\varphi)$ if

$$\delta_\varphi < \left(\|\hat{\varphi}^*(\omega)\|_0 \big/ \|\varphi\|_{L_0^\lambda[a,b]} \right)^{1/\lambda} .$$

Proof

We want to apply Theorem 2 to the proof. Let $t_k = k + \delta_k$. Then we only need to show the following two items.

A. $\{\varphi(t_n - k)\}_k \in l^2$ for any integer n,

B. $C^{-1}\|f\|^2 \le \sum_k |f(t_k)|^2 \le C\|f\|^2$ holds for a constant $C \ge 1$ and for any $f(t) \in V_0(\varphi)$.

The "condition 1" in the theorem implies

$$\hat{\varphi}^*(\omega) \in L^\infty[0,2\pi] \subset L^2[0,2\pi] .$$

Hence

$$\sum_k |\varphi(k)|^2 \le \frac{1}{2\pi} \|\hat{\varphi}^*(\omega)\|_\pi < \infty .$$

Since

$$\begin{aligned} &\left(\sum_k |\varphi(t_n - k)|^2\right)^{1/2} \\ &\le \left(\sum_k |\varphi(n-k)|^2\right)^{1/2} + \left(\sum_k |\varphi(t_n - k) - \varphi(n-k)|^2\right)^{1/2} \\ &\le \left(\sum_k |\varphi(k)|^2\right)^{1/2} + \|\varphi\|_{L_0^\lambda[a,b]} \sup_k |\delta_k|^\lambda \end{aligned}$$

we derive $\{\varphi(t_n - k)\}_k \in l^2$ due to $\varphi(t) \in L_0^\lambda[a,b]$. It is exactly the ``item A".

On the other hand, if we can show that there is a positive number $\theta < 1$ such that

$$\sum_k |f(t_k) - f(k)|^2 \le \theta^2 \sum_k |f(k)|^2$$

holds for any $f(t) \in V_0(\varphi)$, then

$$(1-\theta)^2\sum_k |f(k)|^2 \le \sum_k |f(t_k)|^2 \le (1+\theta)^2\sum_k |f(k)|^2$$

The “condition 1” also implies that there is an $S(t)$ such that (see Chen-Itoh[9])

$$f(t)=\sum_n f(n)S(t-n)\ .$$

This together with the theorem in the former section implies that

$$C^{-1}\|f\|^2 \le \sum_k |f(k)|^2 \le C\|f\|^2$$

holds for some $C \ge 1$ and for any $f(t)\in V_0(\varphi)$. Therefore the ``item B" is demonstrated. In order to show our assumption, we let

$$f(t)=\sum_n c_k\varphi(t-k),$$

and let

$$\begin{aligned}\Delta &= \sum_k |f(t_k)-f(k)|^2\\ &= \sum_k \left|\sum_l c_l(\varphi(t_k-l)-\varphi(k-l))\right|^2\\ &= \sum_k\sum_{l,m} c_l c_m(\varphi(t_k-l)-\varphi(k-l))(\varphi(t_k-m)-\varphi(k-m))\\ &= \sum_{l,m} c_l c_m \sum_k(\varphi(t_k-l)-\varphi(k-l))(\varphi(t_k-m)-\varphi(k-m))\end{aligned}$$

Take

$$a_{k,l}=\sum_n(\varphi(t_n-k)-\varphi(n-k))(\varphi(t_n-l)-\varphi(n-l)).$$

Then $a_{k,l}=a_{l,k}$ and

$$\begin{aligned}\Delta &= \sum_{k,l} a_{k,l}c_k c_l\\ &\le \sum_{k,l}|a_{k,l}|\left(c_k^2+c_l^2\right)/2\\ &= \sum_{k,l}|a_{k,l}|c_k^2\\ &\le \sum_k c_k^2 \sup_k \sum_l |a_{k,l}|\end{aligned}$$

Furthermore we have

$$\begin{aligned}&\sup_k \sum_l |a_{k,l}|\\ &\le \sup_k \sum_{l,n}\left|(\varphi(\delta_n+n-k)-\varphi(n-k))(\varphi(\delta_n+n-l)-\varphi(n-l))\right|\\ &\le \sup_k \sum_{\alpha,\beta}\left|(\varphi(\delta_{\alpha+k}+\alpha)-\varphi(\alpha))(\varphi(\delta_{\alpha+K}+\beta)-\varphi(\beta))\right|\\ &\le \sup_k \sum_\alpha\left|(\varphi(\delta_{\alpha+k}+\alpha)-\varphi(\alpha))\sum_\beta(\varphi(\delta_{\alpha+K}+\beta)-\varphi(\beta))\right|\\ &\le \left(\|\varphi\|_{L_0^\lambda}[a,b]\sup_\alpha|\delta_\alpha|^\lambda\right)^2\end{aligned}$$

Here we used the index transform $n-k=\alpha$ and $n-l=\beta$. Hence

$$\Delta \le \sum_k c_k^2\left(\|\varphi\|_{L_0^\lambda[a,b]}\sup_k|\delta_k|^\lambda\right)^2$$

Since

$$f(l)=\sum_n c_k\varphi(l-k),$$

we have

$$\hat{f}^*(\omega)=\hat{\varphi}^*(\omega)\sum_k c_k e^{-ik\omega}\ .$$

Therefore

$$2\pi\|\hat{\varphi}^*\|_0^2\sum_k c_k^2\le\left\|\hat{\varphi}^*(\omega)\sum_k c_k e^{-ik\omega}\right\|_\pi^2=\left\|\hat{f}^*(\omega)\right\|_\pi^2=2\pi\sum_k|f(k)|^2\ .$$

Therefore we only need to show

$$\sum_k c_k^2\left(\|\varphi\|_{L_0^\lambda[a,b]}\sup_k|\delta_k|^\lambda\right)^2\le\|\hat{\varphi}^*\|_0^2\sum_k c_k^2\ .$$

It is exactly the assumption of the theorem.

Remark:

The estimate in our former work is the same to the theorem when $\varphi(t)$ is orthonormal. But this theorem asserts that this estimate of perturbation holds for any generating functions. By the way, the $\{S_n(t)\}_n$ in the theorem is the solution of the equations

$$\left\langle S_m(\omega),\hat{\varphi}^*(\omega)G_\varphi^{-2}(\omega)\sum_k\varphi(n+\delta_n-k)e^{ik\omega}\right\rangle=2\pi\delta_{m,n}\ .$$

4. SHIFT SAMPLING IN SHIFT INVARIANT SUBSPACES

Unfortunately there are some important generating functions $\varphi(t)$'s with $\|\hat{\varphi}^*(\omega)\|_0=0$. An obvious example is the B-spline of order 2, which has been calculated, in our former works. As done by Janssen[13] for Walter Sampling Theorem[18], Chen-Itoh-Shik[6] for irregular sampling theorem, we also deal with it by shift sampling. Then the shift-sampling theorem can be obtained by using Zak-transform $Z_\varphi(\sigma,\omega)$ ($\sigma\in[0,1)$) defined by

$$Z_\varphi(\sigma,\omega)=\sum_n\varphi(\sigma+n)e^{-in\omega}$$

By using this Zak transform instead of $\hat{\varphi}^*(\omega)$, we obtain a shift-sampling version of the aforementioned theorem as the following.

Theorem

Suppose a generating function $\varphi(t)$ of a shift invariant subspace $V_0(\varphi)$ is such that

1. $C^{-1}\le|Z_\varphi(\sigma,\omega)|\le C$ (*a.e.*), for some constant $C\ge1$,
2. $\varphi(t)\in L_\sigma^\lambda[a,b]$.

Then for any $\delta_{\sigma,k}\in[-\delta_{\sigma,\varphi},\delta_{\sigma,\varphi}]\cap[a,b]$, there is a frame $\{S_{\sigma,n}(t)\}_n$ of $V_0(\varphi)$ such that

$$f(t)=\sum_n f(n+\delta_n+\sigma)S_{\sigma,n}(t)$$

holds for any $f(t)\in V_0(\varphi)$ if

$$\delta_{\sigma,k}<\left(\|\hat{\varphi}^*(\omega)\|_0\Big/\|\varphi\|_{L_\sigma^\lambda[a,b]}\right)^{1/\lambda}\ .$$

Remark:

The $\{S_{\sigma,n}(t)\}_n$ in the theorem is the solution of the equations

$$\left\langle S_m(\omega), \hat{\varphi}^*(\omega) G_\varphi^{-2}(\omega) \sum_k \varphi(\sigma + n + \delta_n - k) e^{ik\omega} \right\rangle = 2\pi\delta_{m,n} .$$

5. EXAMPLES TO SHOW THE ALGORITHM

Since Haar function, Daubechies wavelet and Meyer wavelet are all the orthonormal generating functions[20,11,16], the estimate by this theorem is the same to that by our former works (refer to the Remark of the theorem in section 3). We here calculate the B-spline of order 1. We find that the estimate is $\delta_\varphi < 1/2$, which is better than our former estimate $\delta_\varphi < 1/2\sqrt{3}$, and which is shown by Liu-Walter[15] to be the optimal. Unfortunately by now we still can not show that the estimate in our theorem is optimal for a general generating function!

Example[10]: B-spline of order 1 is defined by

$$N_1(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)} .$$

Then $\hat{N}_1^*(\omega) = 1$. Since

$$\|N_1\|_{L_0^1[-1,1]} = 3 ,$$

we derive

$$\delta_{N_1} < 1/3 .$$

However, when $\delta_k \geq 0$ (or $\delta_k \leq 0$) for all integer k,

$$\|N_1\|_{L_0^1[-1,0]} = \|N_1\|_{L_0^1[0,1]} = 2 .$$

Therefore

$$\delta_{N_1} < 1/2 .$$

The $\{S_n(t)\}_n$ is the solution of the equations

$$\left\langle S_m(\omega), \hat{\varphi}^*(\omega) G_\varphi^{-2}(\omega) \left(\delta_n e^{in\omega} + (1-\delta_n) e^{i(n-1)\omega} \right) \right\rangle = 2\pi\delta_{m,n} ,$$

where

$$G_{N_1}(\omega) = \left(1/3 + 2/3\cos^2(\omega/2)\right)^{1/2} ,$$

and

$$\hat{N}_1(\omega) = \left(e^{-i\omega} - 1\right)^2 \Big/ \omega^2 .$$

6. ACKNOWLEDGMENT

This work is partially supported by Japan Society for the Promotion of Sciences.

7. REFERENCES

1. A. Aldroubi and M. Unser, ``Families of wavelet transforms in connection with Shannon Sampling Theory and the Gabor Transform," in *Wavelets: A Tutorial in Theory and Applications,* C. K. Chui (ed.), pp. 509-528, Academic Press, New York, 1992.

2. A. Aldroubi and M. Unser, ``Families of multi-resolution and wavelet spaces with optimal properties," *Numerical Functional Analysis and Optimization*, **14**, pp. 417-446, 1993.
3. A. Aldroubi and M. Unser, ``Sampling procedures in function spaces and asymptotic equivalence with Shannon Sampling Theorem," *Numerical Functional Analysis and Optimization*, **15**, pp. 1-21, 1994.
4. W. Chen, S. Itoh, and J. Shiki, ``Sampling theorem by wavelets for irregularly sampled signals," *IEICE Trans. Fundamentals*, **J79-A**, pp. 1941-1949, 1996.
5. W. Chen, and S. Itoh, ``On irregular sampling in wavelet subspaces," *IEICE Trans. Fundamentals*, **E80-A**, pp. 1299-1307, 1997.
6. W. Chen, S. Itoh, and J. Shiki, ``Irregular sampling theorem for wavelet subspaces," *IEEE Trans. Information Theory*, **44**, pp. 1131-1142, 1998.
7. W. Chen, and S. Itoh, ``Over-sampling Theorem for wavelet subspaces," *IEICE Trans. Fundamentals*, **E81-A**, pp. 131-138, 1998.
8. W. Chen, and S. Itoh, ``Some notes on reconstructing regularly sampled signal by scaling functions with over-sampling property," *IEICE Trans. Fundamentals*, **E81-A**, pp. 2467-2469, 1998.
9. W. Chen, and S. Itoh , ``A Sampling Theorem for Shift Invariant Subspaces," *IEEE Trans. Signal Processing*, **46**, pp. 1841-1844, 1998.
10. C. K. Chui, ``An Introduction to Wavelets," in *Wavelet Analysis and Its Applications,* vol. 1, Academic Press, New York, 1992.
11. I. Daubechies, ``Orthonormal bases of compactly supported wavelets," *Communication on Pure and Applied Math.*, **41**, pp. 909-996, 1988.
12. C. deBoor, C. deVore, and R. Ron, ``the Structure of finitely generated shift invariant spaces in $L^2(R^d)$," *J. Functional Analysis*, **119**, pp. 37-78, 1994.
13. A. J. E. M. Janssen, ``The zak transform and sampling theorem for wavelet subspaces," *IEEE Trans. Signal Processing*, **41**, pp. 3360-3364, 1993.
14. Y. Liu, ``Irregular sampling for spline wavelet subspaces," *IEEE Trans. Information Theory,* **42**, pp. 623-627, 1996.
15. Y. Liu and G. G. Walter, ``Irregular sampling in wavelet subspaces," *J. Fourier Analysis Application*, **2**, pp. 181-189, 1995.
16. Y. Meyer, ``Wavelets and operators," *Cambridge Studies in Advanced Math.,* vol. 37, Cambridge Uni. Press, Cambridge, 1992.
17. M. Unser and A. Aldroubi, ``A general sampling theorem for Non-ideal Acquisition Devices," *IEEE Trans. Signal Processing*, **42**, pp. 2915-2925, 1994.
18. G. G. Walter, ``A Sampling theorem for wavelet subspaces," *IEEE Trans. Information Theory*, **38**, pp. 881-884, 1992.
19. G. G. Walter, ``Wavelet subspaces with an over-sampling property," *Indaga. Math.*, **4**, pp. 499-507, 1993.
20. G. G. Walter, *Wavelets and Orthogonal System with Applications*, CRC Press, Boca Raton, 1994.
21. X. -G. Xia, ``On orthogonal wavelets with the over-sampling property," *J. Fourier Analysis and Applications,* **2**, pp. 193-199, 1994.
22. X. -G. Xia, and Z. Zhang, ``On sampling theorem, wavelets, and wavelet transforms," *IEEE Trans. Signal Processing*, **41**, pp. 3524-3534, 1993.
23. R. M. Young, *Introduction to Non-Harmonic Fourier Series*, Academic Press, New York, 1980.