

Optimal Binary/Quaternary Adaptive Signature Design for Code-Division Multiplexing[†]

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Abstract—When data symbols modulate a signature waveform to move across a channel in the presence of disturbance, the adaptive real/complex signature that maximizes the signal-to-interference-plus-noise ratio (SINR) at the output of the maximum-SINR filter is the minimum-eigenvalue eigenvector of the disturbance autocovariance matrix. In digital communication systems the signature alphabet is finite and digital signature optimization is NP-hard. In this paper, we propose a new adaptive binary signature assignment for CDMA systems based on improved Fincke-Pohst (FP) algorithm that achieve the optimal exhaustive search performance with low complexity. Then, we extend and propose the optimal adaptive signature assignment algorithm with quaternary signature sets. Simulation studies included herein offer performance comparisons with known adaptive signature designs and the theoretical upper bound of the complex/real eigenvector maximizer.

Index Terms—code-division multiplexing, spread-spectrum communications, signature sets, signal-to-interference-plus-noise ratio (SINR).

I. INTRODUCTION

In recent years, there has been renewed interest in optimal signature sets for the growing number of code-division multiplexing applications. In the theoretical context of complex/real-valued signature sets, the early work of Welch [1] on total-squared-correlation (TSC) bounds was followed up by direct minimum-TSC design proposals [2]-[3] and iterative distributed optimization algorithms [4]-[6]. Minimum-mean-square-error (MMSE) minimization is used for the design of signature sets for multiuser systems over multipath channels in [7]. Recently, new bounds on the TSC of *binary* signature sets were found [8] that led to minimum-TSC optimal binary signature set designs for almost all signature lengths and set sizes [8]-[10]. New bounds and optimal designs for minimum TSC quaternary signature sets are derived in [12].

Instead of previous *static* binary/quaternary signature design, we consider the NP-hard problem of finding the *adaptive* binary/quaternary signature in the code division multiplexing system with interference and multipath fading channels, that maximizes the SINR at the output of the maximum-SINR filter. Direct binary quantization of the minimum-eigenvalue eigenvector is proposed in [13]. The rank-2-optimal proposal that constructs binary signature based on two smallest-eigenvalue eigenvectors is described in [14]. The slowest

descent method (SDM) based adaptive binary signature assignment is presented in [15].

In this paper, we propose a new adaptive binary signature assignments based on improved Fincke-Pohst (FP) algorithm. Instead of exhaustive searching, FP algorithm [16] enumerates all vectors lie in a suitable ellipsoid, with complexity of polynomial order in the searching dimension. FP algorithm has been modified and first applied to communication problems of lattice code decoder in [17], then used for MIMO and space-time codes decoder in [18], generally known as sphere decoding algorithm. In this work, we improve and apply FP algorithm in our adaptive finite-alphabet signature design problem. In addition, we extend our proposed adaptive binary signature assignment to adaptive quaternary signature assignment. We prove that, in general, the adaptive quaternary signature assignment with length L can be equivalent to an adaptive binary signature assignment with length $2L$.

II. SYSTEM MODEL

We consider a multiuser CDMA-type environment with processing gain/signature length L , where K signals/users transmit simultaneously in frequency and time. Each user transmits over N resolvable multipath fading channels. Assuming synchronization with the signal of user k , upon carrier demodulation, chip matched-filtering and sampling at the chip rate over a presumed multipath extended data bit period of $L + N - 1$ chips, we obtain the received vector

$$\mathbf{r}(m) = \sqrt{E_k} b_k(m) \mathbf{H}_k \mathbf{s}_k + \mathbf{z}_k + \mathbf{i}_k + \mathbf{n}, \quad m = 0, 1, \dots, (1)$$

where $b_k(m) \in \{\pm 1\}$, $m = 0, 1, 2, \dots$ is the m th data bit; E_k represents transmitted energy per bit period; \mathbf{s}_k is the signature assigned to user k , for binary alphabet $\mathbf{s}_k \in \{\pm 1\}^L$ and for quaternary alphabet $\mathbf{s}_k \in \{\pm 1, \pm j\}^L$ where $j \triangleq \sqrt{-1}$.

$\mathbf{H}_k \in \mathbb{C}^{(L+N-1) \times L}$ is the user k channel matrix of the form

$$\mathbf{H}_k \triangleq \begin{bmatrix} h_{k,1} & 0 & \dots & 0 \\ h_{k,2} & h_{k,1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ h_{k,N} & h_{k,N-1} & & 0 \\ 0 & h_{k,N} & & h_{k,1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & h_{k,N} \end{bmatrix} \quad (2)$$

with entries $h_{k,n}$, $n = 1, \dots, N$, complex Gaussian random variables to model fading phenomena; $\mathbf{z}_k \in \mathbb{C}^{L+N-1}$ rep-

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represents comprehensively multiple-access-interference (MAI) to user k by the other $K - 1$ users, i.e. $\mathbf{z}_k \triangleq \sum_{i=1, i \neq k}^K \sqrt{E_i} b_i(m) \mathbf{H}_i \mathbf{s}_i$; $\mathbf{i}_k \in \mathbb{C}^{L+N-1}$ denotes multipath induced inter-symbol-interference (ISI) to user k by its own signal; and \mathbf{n} is a zero-mean additive Gaussian noise vector with autocorrelation matrix $\sigma^2 \mathbf{I}_{L+N-1}$.

Information bit detection of user k is achieved via linear minimum-mean-square-error (MMSE) filtering (or, equivalently, max-SINR filtering) as follows

$$\hat{b}_k = \text{sgn} \left(\text{Re} \left\{ \mathbf{w}_{MMSE,k}^H \mathbf{r} \right\} \right) \quad (3)$$

where $\mathbf{w}_{MMSE,k} = c \mathbf{R}^{-1} \mathbf{H}_k \mathbf{s}_k \in \mathbb{C}^{L+N-1}$, $\mathbf{R} \triangleq E \{ \mathbf{r} \mathbf{r}^H \}$, $c > 0$, $\{\cdot\}^H$ is the Hermitian operator, $\text{Re}\{\cdot\}$ denotes the real part of a complex number, and $E\{\cdot\}$ represents statistical expectation. The output SINR of the filter $\mathbf{w}_{MMSE,k}$ is given by

$$\begin{aligned} \text{SINR}_{MMSE,k}(\mathbf{s}_k) &= \frac{E \left\{ \left| \mathbf{w}_{MMSE,k}^H (\sqrt{E_k} b_k \mathbf{H}_k \mathbf{s}_k) \right|^2 \right\}}{E \left\{ \left| \mathbf{w}_{MMSE,k}^H (\mathbf{z}_k + \mathbf{i}_k + \mathbf{n}) \right|^2 \right\}} \\ &= E_k \mathbf{s}_k^H \mathbf{H}_k^H \tilde{\mathbf{R}}_k^{-1} \mathbf{H}_k \mathbf{s}_k \end{aligned} \quad (4)$$

where $\tilde{\mathbf{R}}_k \triangleq E \left\{ (\mathbf{z}_k + \mathbf{i}_k + \mathbf{n}) (\mathbf{z}_k + \mathbf{i}_k + \mathbf{n})^H \right\}$ is the autocorrelation matrix of combined channel disturbance. For mathematical convenience we disregard the ISI component in $\tilde{\mathbf{R}}_k$ and approximate $\tilde{\mathbf{R}}_k$ by $\mathbf{R}_k \triangleq E \left\{ (\mathbf{z}_k + \mathbf{n}) (\mathbf{z}_k + \mathbf{n})^H \right\}$. For notational simplicity we define the $L \times L$ matrix

$$\mathbf{Q}_k \triangleq \mathbf{H}_k^H \mathbf{R}_k^{-1} \mathbf{H}_k. \quad (5)$$

Then, the output SINR in (4) can be rewritten as

$$\text{SINR}_{MMSE,k}(\mathbf{s}_k) = E_k \mathbf{s}_k^H \mathbf{Q}_k \mathbf{s}_k. \quad (6)$$

Our objective is to find the signature \mathbf{s}_k that optimizes (maximizes) $\text{SINR}_{MMSE,k}$ of (6), in binary alphabet $\mathbf{s}_k \in \{\pm 1\}^L$ and quaternary alphabet $\mathbf{s}_k \in \{\pm 1, \pm j\}^L$ respectively.

For the case of binary alphabet $\mathbf{s}_k \in \{\pm 1\}^L$, let \mathbf{Q}_{kR} denote the real part of the complex matrix \mathbf{Q}_k , i.e. $\mathbf{Q}_{kR} \triangleq \text{Re}\{\mathbf{Q}_k\}$. The binary signature $\mathbf{s}_k \in \{\pm 1\}^L$ that maximizes $\text{SINR}_{MMSE,k}$ of (6) is equivalent to

$$\begin{aligned} \mathbf{s}_{k,opt}^{(b)} &= \arg \max_{\mathbf{s}_k \in \{\pm 1\}^L} \mathbf{s}_k^T \mathbf{Q}_{kR} \mathbf{s}_k \\ &= \arg \max_{\mathbf{s}_k \in \{\pm 1\}^L} \mathbf{s}_k^T \mathbf{Q}_{kR} \mathbf{s}_k. \end{aligned} \quad (7)$$

The superscript (b) indicates that $\mathbf{s}_{k,opt}^{(b)}$ is binary; $\{\cdot\}^T$ is the transpose operator.

The quaternary signature $\mathbf{s}_k \in \{\pm 1, \pm j\}^L$ that maximizes $\text{SINR}_{MMSE,k}$ of (6) is given by

$$\mathbf{s}_{k,opt}^{(q)} = \arg \max_{\mathbf{s}_k \in \{\pm 1, \pm j\}^L} \mathbf{s}_k^H \mathbf{Q}_k \mathbf{s}_k, \quad (8)$$

the superscript (q) indicates that $\mathbf{s}_{k,opt}^{(q)}$ is quaternary.

III. OPTIMAL BINARY SIGNATURE ASSIGNMENT

Regarding the binary optimization in (7), we first do the follow transformation

$$\begin{aligned} \mathbf{s}_{k,opt}^{(b)} &= \arg \max_{\mathbf{s} \in \{\pm 1\}^L} \mathbf{s}^T \mathbf{Q}_{kR} \mathbf{s} \\ &= \arg \min_{\mathbf{s} \in \{\pm 1\}^L} \mathbf{s}^T (\alpha \mathbf{I} - \mathbf{Q}_{kR}) \mathbf{s}, \end{aligned} \quad (9)$$

where α is a parameter greater than the maximum eigenvalue of the matrix \mathbf{Q}_{kR} and let

$$\mathbf{W} \triangleq \alpha \mathbf{I} - \mathbf{Q}_{kR}. \quad (10)$$

By definition, matrix \mathbf{W} is Hermitian positive definite.

The Cholesky's factorization of matrix \mathbf{W} yields $\mathbf{W} = \mathbf{B}^T \mathbf{B}$, where \mathbf{B} is an upper triangular matrix. Then equation (9) will lead to

$$\begin{aligned} \mathbf{s}_{k,opt}^{(b)} &= \arg \min_{\mathbf{s} \in \{\pm 1\}^L} \mathbf{s}^T \mathbf{W} \mathbf{s} \\ &= \arg \min_{\mathbf{s} \in \{\pm 1\}^L} \|\mathbf{B} \mathbf{s}\|_F^2, \end{aligned} \quad (11)$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

The original Finche-Pohst (FP) algorithm [16] searches through the discrete points \mathbf{s} in the L -dimensional Euclidean space which make the corresponding vectors $\mathbf{z} \triangleq \mathbf{B} \mathbf{s}$ inside a sphere of given radius \sqrt{C} centered at the origin point, i.e. $\|\mathbf{B} \mathbf{s}\|_F^2 = \|\mathbf{z}\|_F^2 \leq C$. This guarantees that only the points that make the corresponding vectors \mathbf{z} within the square distance C from the origin point are considered in the metric minimization.

Compared with the original FP algorithm in [16], we have two main modifications: (i) The original FP algorithm are searching for integer points, i.e. $\mathbf{s} \in \mathbb{Z}^L$, while our searching alphabet is antipodal binary, i.e. $\mathbf{s} \in \{\pm 1\}^L$. Hence, the bounds to calculate each entry are modified, or further tightened to make the algorithm work faster; (ii) According to the binary signature vector obtained by applying the direct sign operator [13] on the real maximum-eigenvalue eigenvector of \mathbf{Q}_{kR} , denoted as "Quantized Binary" or $\mathbf{s}_{quant}^{(b)}$, we can have a very proper square distance setting as

$$C = \mathbf{s}_{quant}^{(b)T} \mathbf{W} \mathbf{s}_{quant}^{(b)}, \quad (12)$$

such that the searching sphere radius is big enough to have at least one signature point fall inside, while in the meantime small enough to have only a few signature points within.

Let b_{ij} , $i = 1, 2, \dots, L$, $j = 1, 2, \dots, L$ denote the entries of the upper triangular matrix \mathbf{B} ; let s_i , $i = 1, 2, \dots, L$ denote the entries of searching signature \mathbf{s} . According to (11), the signature points that make the corresponding vectors $\mathbf{z} = \mathbf{B} \mathbf{s}$ inside the given radius \sqrt{C} can be expressed as

$$\begin{aligned}
\mathbf{s}^T \mathbf{W} \mathbf{s} &= \|\mathbf{B} \mathbf{s}\|_F^2 = \sum_{i=1}^L \left(b_{ii} s_i + \sum_{j=i+1}^L b_{ij} s_j \right)^2 \\
&= \sum_{i=1}^L g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 \\
&= \sum_{i=k}^L g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 + \sum_{i=1}^{k-1} g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 \\
&\leq C
\end{aligned} \tag{13}$$

where $g_{ii} = b_{ii}^2$ and $g_{ij} = b_{ij}/b_{ii}$ for $i = 1, 2, \dots, L$, $j = i+1, \dots, L$. Obviously the second term of (13) is non-negative, hence, to satisfy (13), it is equivalent to consider for every $k = L, L-1, \dots, 1$,

$$\sum_{i=k}^L g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 \leq C. \tag{14}$$

Then, we can start work backwards to find the bounds for signature entries s_L, s_{L-1}, \dots, s_1 one by one.

To evaluate the element s_k of the signature vector \mathbf{s} , referring to (14) we will have

$$\sum_{i=k}^L g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 \leq C, \tag{15}$$

that leads to

$$\begin{aligned}
&\left[-\sqrt{\frac{1}{g_{kk}} \left(C - \sum_{i=k+1}^L g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 \right) - \sum_{j=k+1}^L g_{kj} s_j} \right] \\
&\leq s_k \leq \left[\sqrt{\frac{1}{g_{kk}} \left(C - \sum_{i=k+1}^L g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 \right) - \sum_{j=k+1}^L g_{kj} s_j} \right]
\end{aligned} \tag{16}$$

If we denote

$$\Delta_k = \sum_{j=k+1}^L g_{kj} s_j, \tag{17}$$

$$\begin{aligned}
C_k &= C - \sum_{i=k+1}^L g_{ii} \left(s_i + \sum_{j=i+1}^L g_{ij} s_j \right)^2 \\
&= C_{k+1} - g_{k+1,k+1} (\Delta_{k+1} + s_{k+1})^2,
\end{aligned} \tag{18}$$

and take consideration of $s_k \in \{\pm 1\}$, the bounds for s_k can be expressed as

$$LB_k \leq s_k \leq UB_k, \tag{19}$$

where

$$\begin{aligned}
UB_k &= \min \left(\left\lfloor \sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rfloor, 1 \right), \\
LB_k &= \max \left(\left\lceil -\sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rceil, -1 \right).
\end{aligned} \tag{20}$$

Note that for given radius \sqrt{C} and the matrix \mathbf{W} , the bounds for s_k only depends on the previous evaluated $s_{k+1}, s_{k+2}, \dots, s_L$.

The entries s_L, s_{L-1}, \dots, s_1 are chosen as follows: for a chosen s_L , we can choose a candidate for s_{L-1} satisfying its bounds requirements as in (19) for $k = L-1$. If a candidate for s_{L-1} does not exist, we go back to choose other s_L . Then search for s_{L-1} that meets the bounds requirement for this new s_L . We follow the same procedure to choose s_{L-2} , and so on. When a set of s_L, s_{L-1}, \dots, s_1 is chosen, one signature candidate vector $\mathbf{s} = [s_1, s_2, \dots, s_L]^T$ is obtained. We record all the candidate signature vectors such that the entries satisfy their bounds requirements, and choose the one that gives the smallest $\mathbf{s}^T \mathbf{W} \mathbf{s}$ metric.

Note that this searching procedure will return *all* candidates that satisfy $\mathbf{s}^T \mathbf{W} \mathbf{s} \leq C$ and gives the one with minimum value. There is at least one candidate vector $\mathbf{s}_{quant}^{(b)}$ satisfying all the bounds requirements, since that is how we set C in (12). On the other hand, the optimal exhaustive binary search result $\mathbf{s}_{exhaustive}^{(b)}$ will also fall inside the search bounds, since

$$\mathbf{s}_{exhaustive}^{(b)T} \mathbf{W} \mathbf{s}_{exhaustive}^{(b)} \leq \mathbf{s}_{quant}^{(b)T} \mathbf{W} \mathbf{s}_{quant}^{(b)} = C. \tag{21}$$

Hence, we are guaranteed to find the optimal exhaustive binary search result by the proposed improved FP algorithm.

Algorithm 1 FP Based Binary Signature Design Algorithm

For the binary signature optimization of $\mathbf{s}_{k,opt}^{(b)} = \arg \max_{\mathbf{s} \in \{\pm 1\}^L} \mathbf{s}^T \mathbf{Q}_{kR} \mathbf{s}$:

Step 1: Let $\mathbf{q}_{k,1}$ be the real maximum-eigenvalue eigenvector of \mathbf{Q}_{kR} with eigenvalue $\lambda_{k,1}$. Apply the direct sign operator on $\mathbf{q}_{k,1}$ and obtain $\mathbf{s}_{quant}^{(b)} = \text{sgn}(\mathbf{q}_{k,1})$. Then construct matrix \mathbf{W} as

$$\mathbf{W} \triangleq \alpha \mathbf{I} - \mathbf{Q}_{kR},$$

where α is a parameter set greater than the maximum eigenvalue of the matrix \mathbf{Q}_{kR} , i.e. $\alpha > \lambda_{k,1}$. Set the square distance as $C = \mathbf{s}_{quant}^{(b)T} \mathbf{W} \mathbf{s}_{quant}^{(b)}$.

Step 2: Operate Cholesky's factorization of matrix \mathbf{W} yields $\mathbf{W} = \mathbf{B}^T \mathbf{B}$, where \mathbf{B} is an upper triangular matrix. Let b_{ij} , $i = 1, 2, \dots, L$, $j = 1, 2, \dots, L$ denote the entries of matrix \mathbf{B} . Construct a new upper triangular matrix \mathbf{G} where $g_{ii} = b_{ii}^2$, $g_{ij} = b_{ij}/b_{ii}$, for $i = 1, 2, \dots, L$, $j = i+1, \dots, L$.

Step 3: Search the candidate vector \mathbf{s} with entries s_1, \dots, s_L according to the following procedure.

Input: Matrix \mathbf{W} , Matrix \mathbf{G} and the radius \sqrt{C} obtained by Step 1 and Step 2.

Output: The vector $\mathbf{s}_{min} \in \{\pm 1\}^L$ satisfies $\mathbf{s}^T \mathbf{W} \mathbf{s} \leq C$ and gives the minimum $\mathbf{s}^T \mathbf{W} \mathbf{s}$ metric.

(i) Start from $\Delta_L = 0$, $C_L = C$, $metric = C$, $\mathbf{s}_{min} = \mathbf{s}_{quant}^{(b)}$ and $k = L$.

(ii) Set the upper bound UB_k and the lower bound LB_k as follows

$$\begin{aligned}
UB_k &= \min \left(\left\lfloor \sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rfloor, 1 \right), \\
LB_k &= \max \left(\left\lceil -\sqrt{\frac{C_k}{g_{kk}}} - \Delta_k \right\rceil, -1 \right),
\end{aligned}$$

and $s_k = LB_k - 1$.

- (iii) Set $s_k = s_k + 1$. If $s_k = 0$, set $s_k = 1$. For $s_k \leq UB_k$, go to (v); else go to (iv).
- (iv) If $k = L$, terminate and output s_{min} ; else set $k = k + 1$ and go to (iii).
- (v) For $k = 1$, go to (vi); else set $k = k - 1$, and

$$\Delta_k = \sum_{j=k+1}^L g_{kj} s_j, \quad C_k = C_{k+1} - g_{k+1,k+1} (\Delta_{k+1} + s_{k+1})^2$$

then go to (ii).

- (vi) We get a candidate vector \mathbf{s} that satisfies all the bounds requirements. If $\mathbf{s}^T \mathbf{W} \mathbf{s} \leq \text{metric}$, then update $\mathbf{s}_{min} = \mathbf{s}$ and $\text{metric} = \mathbf{s}^T \mathbf{W} \mathbf{s}$. Go to (iii).

Step 4: Once we get the optimal \mathbf{s}_{min} from Step 3 that returns the minimum $\mathbf{s}^T \mathbf{W} \mathbf{s}$ metric, the optimal adaptive binary signature that maximizes the SINR at the output of MMSE filter is $\mathbf{s}_{k,opt}^{(b)} = \mathbf{s}_{min}$. Note that the solution obtained through this algorithm is guaranteed to be *optimal*, which means the same as exhaustive searching.

IV. OPTIMAL QUATERNARY SIGNATURE ASSIGNMENT

For the quaternary signature optimization of (8), a heuristic approach will be direct quantization signature vector obtained by applying the sign operator on real part and imaginary part of the complex maximum-eigenvalue eigenvector of \mathbf{Q}_k . However, this is a suboptimal approach and the performance is inferior as shown in simulation section.

For a quaternary signature $\mathbf{s} \in \{\pm 1, \pm j\}^L$, a transform is made as

$$\mathbf{s} = \frac{1}{2}(1 - j)\mathbf{c}, \quad (22)$$

such that $\mathbf{c} \in \{-1 - j, -1 + j, 1 - j, 1 + j\}^L$. Note that if the real part and imaginary part of vector \mathbf{c} are denoted as $\mathbf{c}_R = \text{Re}\{\mathbf{c}\}$ and $\mathbf{c}_I = \text{Im}\{\mathbf{c}\}$, the transform will lead to $\mathbf{c}_R \in \{\pm 1\}^L$ and $\mathbf{c}_I \in \{\pm 1\}^L$, two binary antipodal sequences.

By definition, \mathbf{Q}_k in (8) is Hermitian positive definite. So we can operate on the matrix of \mathbf{Q}_k Cholesky decomposition

$$\mathbf{Q}_k = \mathbf{U}^H \mathbf{U}, \quad (23)$$

where \mathbf{U} is an upper triangular matrix. Then

$$\mathbf{s}^H \mathbf{Q}_k \mathbf{s} = \left(\frac{1}{2}(1 - j)\mathbf{c} \right)^H \mathbf{Q}_k \left(\frac{1}{2}(1 - j)\mathbf{c} \right) = \frac{1}{2} \|\mathbf{U}\mathbf{c}\|_F^2. \quad (24)$$

Define $\mathbf{y} \triangleq \mathbf{U}\mathbf{c}$ and let $\mathbf{y}_R = \text{Re}\{\mathbf{y}\}$ and $\mathbf{y}_I = \text{Im}\{\mathbf{y}\}$, $\mathbf{U}_R = \text{Re}\{\mathbf{U}\}$ and $\mathbf{U}_I = \text{Im}\{\mathbf{U}\}$. Then, it is easy to obtain the following equation

$$\begin{bmatrix} \mathbf{y}_R \\ \mathbf{y}_I \end{bmatrix} = \begin{bmatrix} \mathbf{U}_R & -\mathbf{U}_I \\ \mathbf{U}_I & \mathbf{U}_R \end{bmatrix} \begin{bmatrix} \mathbf{c}_R \\ \mathbf{c}_I \end{bmatrix}. \quad (25)$$

Hence, combining equations (24) and (25) will lead to

$$\begin{aligned} \mathbf{s}^H \mathbf{Q}_k \mathbf{s} &= \frac{1}{2} \left\| \begin{bmatrix} \mathbf{U}_R & -\mathbf{U}_I \\ \mathbf{U}_I & \mathbf{U}_R \end{bmatrix} \begin{bmatrix} \mathbf{c}_R \\ \mathbf{c}_I \end{bmatrix} \right\|_F^2 \\ &= \underbrace{\begin{bmatrix} \mathbf{c}_R \\ \mathbf{c}_I \end{bmatrix}^T}_{\bar{\mathbf{c}}^T} \underbrace{\frac{1}{2} \begin{bmatrix} \mathbf{U}_R & -\mathbf{U}_I \\ \mathbf{U}_I & \mathbf{U}_R \end{bmatrix}^T \begin{bmatrix} \mathbf{U}_R & -\mathbf{U}_I \\ \mathbf{U}_I & \mathbf{U}_R \end{bmatrix}}_{\bar{\mathbf{Q}}_{kR}} \underbrace{\begin{bmatrix} \mathbf{c}_R \\ \mathbf{c}_I \end{bmatrix}}_{\bar{\mathbf{c}}}, \end{aligned} \quad (26)$$

where $\bar{\mathbf{c}} \triangleq \begin{bmatrix} \mathbf{c}_R \\ \mathbf{c}_I \end{bmatrix} \in \{\pm 1\}^{2L}$ is a binary signature with length $2L$. Note that $\mathbf{c} = \mathbf{c}_R + j\mathbf{c}_I$ is complex signature with length L .

Now, as shown in (26), the quaternary signature optimization with length L in (8) is transformed into the following binary signature optimization problem with length $2L$

$$\bar{\mathbf{c}}_{opt}^{(b)} = \arg \max_{\bar{\mathbf{c}} \in \{\pm 1\}^{2L}} \bar{\mathbf{c}}^T \bar{\mathbf{Q}}_{kR} \bar{\mathbf{c}}. \quad (27)$$

After we get the optimal binary sequence $\bar{\mathbf{c}}_{opt}^{(b)}$ of length $2L$,

split $\bar{\mathbf{c}}_{opt}^{(b)}$ into $\bar{\mathbf{c}}_{opt}^{(b)} = \begin{bmatrix} \mathbf{c}_{R,opt}^{(b)} \\ \mathbf{c}_{I,opt}^{(b)} \end{bmatrix}$, where $\mathbf{c}_{R,opt}^{(b)}$ and $\mathbf{c}_{I,opt}^{(b)}$

are binary sequences in length L , i.e. $\mathbf{c}_{R,opt}^{(b)} \in \{\pm 1\}^L$ and $\mathbf{c}_{I,opt}^{(b)} \in \{\pm 1\}^L$. Then, the optimal quaternary signature can be constructed as

$$\mathbf{s}_{opt}^{(q)} = \frac{1}{2}(1 - j) \left(\mathbf{c}_{R,opt}^{(b)} + j\mathbf{c}_{I,opt}^{(b)} \right). \quad (28)$$

We summarize the quaternary optimization problem of (8) and propose the following algorithm.

Algorithm 2

FP Based Quaternary Signature Design Algorithm

For the quaternary signature optimization of $\mathbf{s}_{k,opt}^{(q)} = \arg \max_{\mathbf{s} \in \{\pm 1, \pm j\}^L} \mathbf{s}^H \mathbf{Q}_k \mathbf{s}$:

Step 1: We operate on the complex matrix of \mathbf{Q}_k Cholesky decomposition

$$\mathbf{Q}_k = \mathbf{U}^H \mathbf{U}.$$

Let $\mathbf{U}_R = \text{Re}\{\mathbf{U}\}$ and $\mathbf{U}_I = \text{Im}\{\mathbf{U}\}$. Construct real matrix $\bar{\mathbf{Q}}_{kR}$ as follows

$$\bar{\mathbf{Q}}_{kR} = \frac{1}{2} \begin{bmatrix} \mathbf{U}_R & -\mathbf{U}_I \\ \mathbf{U}_I & \mathbf{U}_R \end{bmatrix}^T \begin{bmatrix} \mathbf{U}_R & -\mathbf{U}_I \\ \mathbf{U}_I & \mathbf{U}_R \end{bmatrix}.$$

Step 2: Solve the following binary signature optimization problem with signature length $2L$ based on *FP Based Binary Signature Design Algorithm*

$$\bar{\mathbf{c}}_{opt}^{(b)} = \arg \max_{\bar{\mathbf{c}} \in \{\pm 1\}^{2L}} \bar{\mathbf{c}}^T \bar{\mathbf{Q}}_{kR} \bar{\mathbf{c}}.$$

Step 3: Split $\bar{\mathbf{c}}_{opt}^{(b)} = \begin{bmatrix} \mathbf{c}_{R,opt}^{(b)} \\ \mathbf{c}_{I,opt}^{(b)} \end{bmatrix}$, where $\mathbf{c}_{R,opt}^{(b)}$ and $\mathbf{c}_{I,opt}^{(b)}$ are binary sequences in length L . Then, the optimal quaternary signature can be constructed as

$$\mathbf{s}_{opt}^{(q)} = \frac{1}{2}(1 - j) \left(\mathbf{c}_{R,opt}^{(b)} + j\mathbf{c}_{I,opt}^{(b)} \right).$$

According to the same analysis as in the previous section, FP Based Quaternary Signature Design Algorithm is also

guaranteed to find the optimal exhaustive searching result with complexity order of polynomial in the signature length L .

We note that by using the same quaternary binary equivalence procedure, we can also extend our previous proposed SDM Based Binary Signature Design Algorithm in [15] to solve the quaternary signature optimization of (8). We denote it as SDM Based Quaternary Signature Design Algorithm which performance comparisons will follow in the simulation studies section.

V. SIMULATION STUDIES

We first compare performance of adaptive *binary* signature assignment of the following benchmarks: (i) "Real max-EV": The real maximum-eigenvalue eigenvector of $\mathbf{Q}_{kR} = \text{Re}\{\mathbf{Q}_k\}$; (ii) "Exhaustive Binary": The binary signature assigned by exhaustive search; (iii) "Quantized Binary": The binary signature obtained by applying the sign operator on the real maximum-eigenvalue eigenvector of \mathbf{Q}_{kR} [13]; (iv) "Rank2 Binary": The adaptive rank-2 binary signature design [14]; (v) "SDM Based Binary Algorithm": The adaptive binary signature assignment in [15] based on the top $P \geq 2$ real maximum-eigenvalue eigenvectors; (v) "FP Based Binary Algorithm": The optimal binary signature assignment proposed in this paper.

We consider a code-division multiplexing multipath fading system model with $L = 16$ and $N = 3$. The signal power of the user of interest is set to $E_1 = 10\text{dB}$, while E_2, E_3, \dots, E_K are uniformly spaced between 8dB and 11dB . The interfering spreading signatures are randomly generated. For comparison purposes, we evaluate the SINR loss, the difference between SINR of the optimal real signature (Real max-EV) and other adaptive binary signature assignment algorithms. The results that we present are averages over 1000 randomly generated interferences and channel realizations.

In Fig. 1, we plot the SINR loss as a function of the number of interferences. We observe that SDM based binary algorithm and FP based binary algorithm offer superior performance than direct quantized binary and rank 2 assignments. Furthermore, FP based binary algorithm actually achieves exactly the optimal exhaustive binary search assignment as we expected.

Then we investigate the multiuser binary signature assignment in a sequential user-after-user manner based on various adaptive binary signature assignments. In such an approach, each user's spreading signature is updated one after the other. Several multiuser adaptation cycles are carried out until numerical convergence is observed. In Fig. 2, for a total of $K = 8$ users, we plot the SINR loss of one user of interest as a function of multiuser adaptation cycle. Still, SDM based binary algorithm and FP based binary algorithm offer superior performance than the direct quantized binary and rank2 assignments. FP based binary algorithm achieves exactly the optimal exhaustive binary search assignment.

We repeat our studies for adaptive *quaternary* signature assignment algorithms: (i) "Complex max-EV": The complex maximum-eigenvalue eigenvector of \mathbf{Q}_k ; (ii) "Exhaustive Quaternary": The quaternary signature assigned by exhaustive

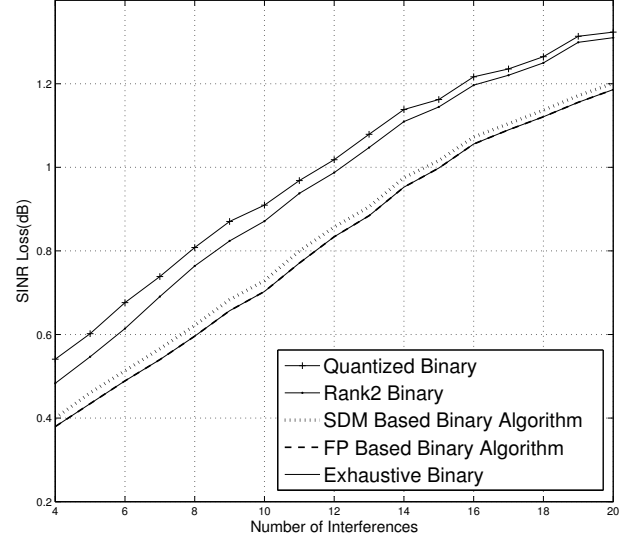


Fig. 1. SINR Loss of various adaptive binary signature assignments versus number of interferences ($L=16$).

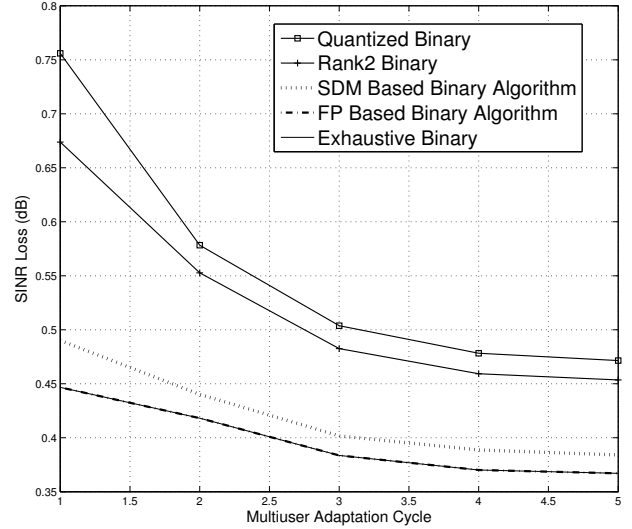


Fig. 2. SINR Loss of various adaptive binary signature assignments versus multiuser adaptation cycle ($L=16$, $K=8$).

search; (iii) "Quantized Quaternary": The quaternary signature obtained by applying sign operator on real part and imaginary part of the complex maximum-eigenvalue eigenvector of \mathbf{Q}_k , mentioned in the beginning of section IV; (iv) "SDM Based Quaternary Algorithm": The SDM based quaternary signature design based on the quaternary-binary equivalence procedure and the application of SDM based binary signature assignment in [15]; (v) "FP Based Quaternary Algorithm": The quaternary signature design algorithm proposed in this paper as *FP Based Quaternary Signature Design Algorithm*. The SINR loss for

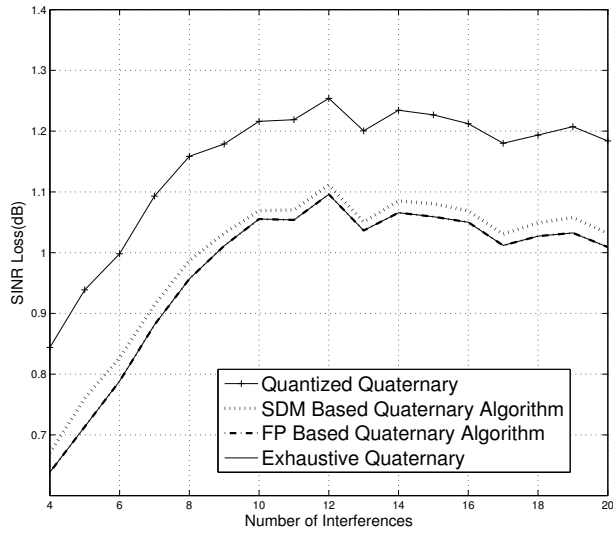


Fig. 3. SINR Loss of various adaptive quaternary signature assignments versus number of interferences ($L=8$).

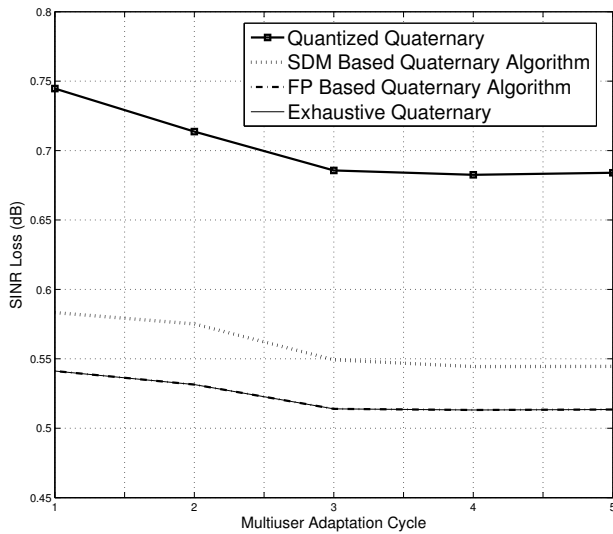


Fig. 4. SINR Loss of various adaptive quaternary signature assignments versus multiuser adaptation cycle ($L=8$, $K=4$).

quaternary assignments are the difference between SINR of the optimal complex signature (Complex max-EV) and other adaptive quaternary assignment algorithms.

We plot the SINR loss as a function of the number of interferences in Fig. 3, and as a function of multiuser adaptation cycle in Fig. 4. We obtain the same results as previous adaptive binary simulations.

VI. CONCLUSIONS

We propose a new adaptive binary signature assignments based on improved FP algorithm, that returns the optimal ex-

haustive searching result with low complexity. In addition, we extend to adaptive quaternary signature assignments and prove that, the adaptive quaternary signature assignment with length L can be equivalent to an adaptive binary signature assignment with length $2L$. Simulation studies show the comparisons with the optimal FP based binary/quaternary signature assignment and previous existing signature assignments.

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