

Optimal Upward Scaling of Minimum-TSC Binary Signature Sets

Lili Wei, *Member, IEEE*, and Wen Chen, *Senior Member, IEEE*

Abstract—We consider upward scaling an overloaded minimum-TSC binary signature set and propose an optimal solution based on improved sphere decoding algorithm. Instead of previous suboptimal approach, we are guaranteed to find the optimal maximum-likelihood (ML) searching result with low complexity.

Index Terms—Binary sequences, code-division multiple-access (CDMA), signal design, total squared correlation, Welch bound.

I. INTRODUCTION

IN multiuser communication systems that follow the code-division multiplexing paradigm, multiple signals are transmitted simultaneously in time and frequency. Each signal, potentially associated with a distinct user, is assigned an individual signature (spreading code). A fundamental measure of the quality of the code-division communication link is the total squared correlation (TSC) [1] over the set of assigned signatures. For a K -signal system with signature length L , if the signature set is denoted by $S = \{s_1, s_2, \dots, s_K\}$, $\|s_i\|^2 = L$, $i = 1, 2, \dots, K$, then the TSC of the signature set S is defined as the sum of the squared magnitudes of all inner products between signatures,

$$TSC(S) \triangleq \sum_{i=1}^K \sum_{j=1}^K |s_i^H s_j|^2, \quad (1)$$

where $(\cdot)^H$ denotes the conjugate transpose operation. For real/complex-valued signature sets, TSC is bounded from the “Welch Bound” $TSC(S) \geq KL \max\{K, L\}$ and signature sets that satisfy this bound with equality are called Welch-bound-equality (WBE) sets [1] [2]. Algorithms and studies for the design of complex or real WBE signature sets can be found in [3]–[6].

In digital transmission systems, it is necessary to have finite-alphabet signature sets. Although the Welch bound is always achievable for real/complex-valued signature sets, this is not the case in general for binary antipodal signature sets. Hence, findings in [1]–[6] constitute only pertinent performance bounds for digital communication systems with digital signatures. In [7], new bounds on the TSC of *binary* signature sets were presented that lead to minimum-TSC optimal binary signature set designs for almost all signature lengths and

set sizes [7]–[9]. The user capacity of minimum and non-minimum-TSC binary sets was identified and compared in [10]. A procedure to find minimum-TSC binary signature sets with low cross-correlation spectrum was presented in [11].

The technical problem we consider in this manuscript is upward scaling of an overloaded ($K > L$) min-TSC binary set. Subsets of underloaded ($K \leq L$) signatures maintain TSC optimality and signatures can be returned and reassigned without loss of optimality. This is not the case unfortunately, in general, given a min-TSC overloaded set (K, L) by [7]. Addition of a signature, for example, may require complete redesign/reassignment of the $(K + 1, L)$ set to maintain TSC optimality. Previous suboptimal solution on this problem has been approached in [12] based on slowest decent method.

In this manuscript, we are relocating this problem with improved sphere decoding (SD) algorithm. The original SD algorithm was proposed in [13] as Fincke-Pohst method, first applied to communication system of lattice code decoder in [14] and then used for space-time decoding in [15]. Instead of exhaustive maximum-likelihood (ML) searching, SD algorithm, with complexity of polynomial order in the processing gain L [13], considers only a small set of vectors within a given sphere rather than all possible transmitted signal vectors. With a proper initial searching radius setting, our proposed improved SD algorithm is guaranteed to find the optimal ML result with low complexity instead of previous suboptimal solutions.

II. SYSTEM MODEL

We consider a code division multiplexing system with code length L and $K \geq L$ signals (overloaded). The K signals utilize a minimum TSC optimal binary signature set S designed according to [7], $S = \{s_1, s_2, \dots, s_K\}$, $s_i \in \{\pm 1\}^L$, $i = 1, \dots, K$. The TSC lower bound of this binary antipodal signature sets for overloaded ($K \geq L$) systems are given in Table II in [7].

When a new signal enters this system with signature $s_{K+1} \in \{\pm 1\}^L$, the TSC of the $K + 1$ signatures, given the signatures of the K preexisting signals, is

$$\begin{aligned} TSC_{K+1|K} &= \sum_{i=1}^{K+1} \sum_{j=1}^{K+1} |s_i^T s_j|^2 \\ &= \sum_{i=1}^K \sum_{j=1}^K |s_i^T s_j|^2 + |s_{K+1}^T s_{K+1}|^2 + 2 \sum_{i=1}^K |s_{K+1}^T s_i|^2 \\ &= TSC_K + L^2 + 2s_{K+1}^T \left(\sum_{i=1}^K s_i s_i^T \right) s_{K+1}, \end{aligned} \quad (2)$$

where TSC_K denotes the TSC of the K preexisting signals in the system that utilizes a minimum TSC binary signature

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The authors are with the Department of Electronic Engineering, Shanghai Jiao Tong University, Shanghai, China, and SKL for ISN, Xidian University, China (e-mail: {liliwei, wencheng}@sjtu.edu.cn).

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set. If we denote the autocorrelation matrix of the preexisting K signatures by

$$\mathbf{R}_K = \sum_{i=1}^K \mathbf{s}_i \mathbf{s}_i^T, \quad (3)$$

Equation (2) shows that conditional minimization of $TSC_{K+1|K}$ with respect to \mathbf{s}_{K+1} for fixed (min-TSC-valued) TSC_K reduces to

$$\mathbf{s}_{K+1} = \arg \min_{\mathbf{s} \in \{\pm 1\}^L} \mathbf{s}^T \mathbf{R}_K \mathbf{s}. \quad (4)$$

The optimal ML solution by exhaustive search over all 2^L vectors in $\{\pm 1\}^L$ to find the one that minimizes (4) is, of course, unacceptable computationally even for moderate values of L . The work of [12] has proposed a suboptimal solution based on slowest decent method. In the following section, we are going to solve this problem with improved SD algorithm which gives us the optimal ML solution with low complexity.

III. OPTIMAL SIGNATURE ASSIGNMENT

Let the Cholesky's factorization of the autocorrelation matrix \mathbf{R}_K yields $\mathbf{R}_K = \mathbf{U}^T \mathbf{U}$, where \mathbf{U} is an upper triangular matrix. Then

$$\mathbf{s}_{K+1} = \arg \min_{\mathbf{s} \in \{\pm 1\}^L} \mathbf{s}^T \mathbf{R}_K \mathbf{s} = \arg \min_{\mathbf{s} \in \{\pm 1\}^L} \|\mathbf{U} \mathbf{s}\|_F^2, \quad (5)$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

The original SD decoding algorithm [13]-[14] searches through the discrete points \mathbf{s} in the L -dimensional Euclidean space which make the corresponding vectors $\mathbf{z} \triangleq \mathbf{U} \mathbf{s}$ inside a sphere of given radius \sqrt{C} centered at the origin point, i.e. $\|\mathbf{U} \mathbf{s}\|_F^2 = \|\mathbf{z}\|_F^2 \leq C$. This guarantees that only the points that make the corresponding vectors \mathbf{z} within the square distance C from the origin point are considered in the metric minimization.

Compared with the original SD algorithm, we have two main modifications: (i) The original SD algorithm are searching for integer points, i.e. $\mathbf{s} \in \mathbb{Z}^L$, while our signature searching alphabet is antipodal binary, i.e. $\mathbf{s} \in \{\pm 1\}^L$. Hence, the bounds to calculate each entry of the optimal signature are modified, or further tightened, according to our binary searching alphabet to make the algorithm work faster; (ii) According to the binary signature vector obtained by applying the direct sign operator on the real minimum-eigenvalue eigenvector of \mathbf{R}_K , denoted as $\mathbf{s}_{quant}^{(b)}$, we can have a very proper square distance setting as

$$C = \mathbf{s}_{quant}^{(b)T} \mathbf{R}_K \mathbf{s}_{quant}^{(b)}, \quad (6)$$

such that the searching sphere radius is big enough to have at least one signature point fall inside, while in the meantime small enough to have only a few signature points within. As we can have this appropriate radius setting, we calculate the $\mathbf{s}^T \mathbf{R}_K \mathbf{s}$ metric for every candidate vector \mathbf{s} that satisfies $\|\mathbf{U} \mathbf{s}\|_F^2 \leq C$, such that the optimal signature assignment with minimum $\mathbf{s}^T \mathbf{R}_K \mathbf{s}$ metric is obtained from the improved SD algorithm directly.

Since the radius is fixed for our improved SD algorithm, the complexity uncertainty due to the radius update, which means

that the radius need to be expanded if no points found in the sphere and the radius need to be reduced if too many points within the sphere, is not a question in this optimization.

Let u_{ij} , $i, j = 1, \dots, L$, denote the entries of matrix \mathbf{U} in (5). Then we are searching among $\mathbf{s} \in \{\pm 1\}^L$ such that

$$\begin{aligned} \mathbf{s}^T \mathbf{R}_K \mathbf{s} &= \|\mathbf{U} \mathbf{s}\|_F^2 = \sum_{i=1}^L \left(u_{ii} s_i + \sum_{j=i+1}^L u_{ij} s_j \right)^2 \\ &= \sum_{i=1}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij} s_j \right)^2 \leq C \end{aligned} \quad (7)$$

where $\mathbf{s} = [s_1, s_2, \dots, s_L]^T$, $q_{ii} = u_{ii}^2$ for $i = 1, \dots, L$ and $q_{ij} = u_{ij}/u_{ii}$ for $i = 1, \dots, L$, $j = i+1, \dots, L$.

We can start work backwards to find the entries s_L, s_{L-1}, \dots, s_1 one by one.

Step 1: We begin to evaluate the last integer element s_L . Referring to (7) and consider $q_{LL} s_L^2 \leq C$. For $s_L \in \{\pm 1\}$, s_L can be chosen arbitrarily.

Step 2: Referring to (7) again, a candidate value for s_{L-1} is chosen satisfying the following

$$q_{LL} s_L^2 + q_{L-1,L-1} (s_{L-1} + q_{L-1,L} s_L)^2 \leq C \quad (8)$$

which lead to

$$\left\lceil -\sqrt{\frac{C - q_{LL} s_L^2}{q_{L-1,L-1}}} - q_{L-1,L} s_L \right\rceil \leq s_{L-1} \leq \left\lfloor \sqrt{\frac{C - q_{LL} s_L^2}{q_{L-1,L-1}}} - q_{L-1,L} s_L \right\rfloor, \quad (9)$$

where $\lceil x \rceil$ is the smallest integer greater than x and $\lfloor x \rfloor$ is the greatest integer smaller than x . If we denote $\Delta_{L-1} = q_{L-1,L} s_L$ and $C_{L-1} = C - q_{LL} s_L^2$, and consider $s_{L-1} \in \{\pm 1\}$, the bounds for s_{L-1} can be expressed as

$$LB_{L-1} \leq s_{L-1} \leq UB_{L-1}, \quad (10)$$

$$UB_{L-1} = \min \left(\left\lfloor \sqrt{\frac{C_{L-1}}{q_{L-1,L-1}}} - \Delta_{L-1} \right\rfloor, 1 \right) \quad (11)$$

$$LB_{L-1} = \max \left(\left\lceil -\sqrt{\frac{C_{L-1}}{q_{L-1,L-1}}} - \Delta_{L-1} \right\rceil, -1 \right). \quad (12)$$

We can see that given radius \sqrt{C} and the matrix \mathbf{R}_K , the bounds for s_{L-1} only depends on the previous evaluated s_L , and is not correlated with $s_{L-2}, s_{L-3}, \dots, s_1$.

In a similar fashion, we can proceed for s_{L-2} , and so on.

Step L-k+1: For the component of s_k , referring to (7) and consider

$$\sum_{i=k}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij} s_j \right)^2 \leq C \quad (13)$$

will lead to

$$\begin{aligned} &\left\lceil -\sqrt{\frac{1}{q_{kk}} \left(C - \sum_{i=k+1}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij} s_j \right)^2 \right)} - \sum_{j=k+1}^L q_{kj} s_j \right\rceil \\ &\leq s_k \leq \left\lfloor \sqrt{\frac{1}{q_{kk}} \left(C - \sum_{i=k+1}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij} s_j \right)^2 \right)} - \sum_{j=k+1}^L q_{kj} s_j \right\rfloor. \end{aligned} \quad (14)$$

If we denote

$$\Delta_k = \sum_{j=k+1}^L q_{kj}s_j, \quad C_k = C - \sum_{i=k+1}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij}s_j \right)^2 \quad (15)$$

and take consideration of $s_k \in \{\pm 1\}$, the bounds for s_k can be expressed as

$$LB_k \leq s_k \leq UB_k, \quad (16)$$

$$UB_k = \min \left(\left\lfloor \sqrt{\frac{C_k}{q_{kk}}} - \Delta_k \right\rfloor, 1 \right), \quad (17)$$

$$LB_k = \max \left(\left\lceil -\sqrt{\frac{C_k}{q_{kk}}} - \Delta_k \right\rceil, -1 \right). \quad (18)$$

Note that for given radius \sqrt{C} and the matrix \mathbf{R}_K , the bounds for s_k only depends on the previous evaluated $s_{k+1}, s_{k+2}, \dots, s_L$.

Step L: To evaluate the range of integer component s_1 , referring to (7) and consider

$$\sum_{i=1}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij}s_j \right)^2 \leq C \quad (19)$$

will lead to

$$\left[-\sqrt{\frac{1}{q_{11}} \left(C - \sum_{i=2}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij}s_j \right)^2 \right) - \sum_{j=2}^L q_{1j}s_j} \right] \leq s_1 \leq \left[\sqrt{\frac{1}{q_{11}} \left(C - \sum_{i=2}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij}s_j \right)^2 \right) - \sum_{j=2}^L q_{1j}s_j} \right]. \quad (20)$$

If we denote

$$\Delta_1 = \sum_{j=2}^L q_{1j}s_j, \quad C_1 = C - \sum_{i=2}^L q_{ii} \left(s_i + \sum_{j=i+1}^L q_{ij}s_j \right)^2, \quad (21)$$

and take consideration of $s_1 \in \{\pm 1\}$, the bounds for s_1 can be expressed as

$$LB_1 \leq s_1 \leq UB_1, \quad (22)$$

$$UB_1 = \min \left(\left\lfloor \sqrt{\frac{C_1}{q_{11}}} - \Delta_1 \right\rfloor, 1 \right), \quad (23)$$

$$LB_1 = \max \left(\left\lceil -\sqrt{\frac{C_1}{q_{11}}} - \Delta_1 \right\rceil, -1 \right). \quad (24)$$

In practice, C_{L-1}, \dots, C_1 can be updated recursively by the following equations with initial settings $\Delta_L = 0, C_L = C$,

$$\begin{cases} \Delta_k &= \sum_{j=k+1}^L q_{kj}s_j, \\ C_{k-1} &= C_k - q_{kk} (\Delta_k + s_k)^2. \end{cases} \quad (25)$$

The entries s_L, s_{L-1}, \dots, s_1 are chosen as follows: for a chosen s_L , we can choose a candidate for s_{L-1} satisfying the bounds (10)-(12). If such s_{L-1} does not exist, we go back and choose other s_L . Then search for s_{L-1} that meets the bounds (10)-(12) for the given s_L . If s_L and s_{L-1} are chosen, we follow the same procedure to choose s_{L-2} , and so on. When a set of s_L, s_{L-1}, \dots, s_1 is chosen and satisfies all corresponding bounds requirements, one signature candidate

vector $\mathbf{s} = [s_1, s_2, \dots, s_L]^T$ is obtained. We choose the one among all candidates that gives the smallest $\mathbf{s}^T \mathbf{R}_K \mathbf{s}$ metric.

Note that this searching procedure will go through *all* candidates that satisfy $\mathbf{s}^T \mathbf{R}_K \mathbf{s} \leq C$ and gives the one with minimum value. There is at least one candidate vector $\mathbf{s}_{quant}^{(b)}$ such that its entries satisfy all the bounds requirements, since that is how we set the radius value in (6). On the other hand, the ML exhaustive binary search result $\mathbf{s}_{ML}^{(b)}$ that returns the minimum metric will also fall inside the search bounds, since

$$\mathbf{s}_{ML}^{(b)T} \mathbf{R}_K \mathbf{s}_{ML}^{(b)} \leq \mathbf{s}_{quant}^{(b)T} \mathbf{R}_K \mathbf{s}_{quant}^{(b)} = C. \quad (26)$$

Hence, we are guaranteed to find the optimal ML exhaustive binary search result by the proposed improved SD algorithm.

Regarding the computational cost for the proposed improved SD based algorithm, first, eigen-decomposition needed for parameter setting of the square distance C in (6), will have complexity cost $O(L^3)$. In addition, for the improved SD algorithm with fixed square distance C , [13] gives a complexity analysis and shows that the number of arithmetic operations is at most

$$\frac{1}{6} (2L^3 + 3L^2 - 5L) + \frac{1}{2} (L^2 + 12L - 7) \times \left((2\lfloor \sqrt{Ct} \rfloor + 1) \binom{\lfloor 4Ct \rfloor + L - 1}{\lfloor 4Ct \rfloor} + 1 \right), \quad (27)$$

where t^{-1} is the lower bound for the entries $u_{11}^2, u_{22}^2, \dots, u_{LL}^2$ of matrix \mathbf{U} .

Hence, the total computational cost for the proposed improved SD based algorithm still have polynomial complexity. In the literature of [12], the binary signature assignment obtained on slowest descent method (SDM) has been proposed. Compared with SDM algorithm, the proposed improved SD algorithm has additional computational cost of (27). However, at the expense of this additional computational cost, the proposed improved SD based algorithm is guaranteed to get the optimal ML exhaustive searching result.

IV. EXPERIMENTAL STUDIES

We consider a code-division multiplexing system ($K + 1, L = 16$) for $K = 16$ up to 31 where each (K, L) signature set is optimally min-TSC designed by [7]. We compare the TSC performance of: (i) TSC_{bound} : the $(K + 1, L)$ TSC lower bound of [7]; (ii) TSC_{SDM} : the $K + 1$ signature added by the previous suboptimal approach of [12] based on slowest descent method (SDM); (iii) TSC_{SD} : the $K + 1$ signature added by proposed improved SD algorithm in this manuscript; (iv) TSC_{ML} : the $K + 1$ signature added by the ML exhaustive searching. For comparison purpose, we evaluate the TSC difference with the lower bound of [7], i.e. $TSC_{SDM} - TSC_{bound}$, $TSC_{SD} - TSC_{bound}$, $TSC_{ML} - TSC_{bound}$, and plot in Fig. 1.

The comparison with theoretical minimum TSC bounds is very favorable. Frequently, the resulting sequence set is absolutely TSC optimal. From the detailed simulation data analysis, we find that the results of the proposed improved SD algorithm always reaches ML solution as we expected, and is superior to the previous suboptimal SDM algorithm. We demonstrate this in Table 1 with some typical simulation

TABLE I
TSC COMPARISON OF SDM, SD AND ML

$K + 1$	TSC with SDM	TSC with SD	TSC with ML
19	6544	6400	6400
23	9104	9088	9088
27	12304	12288	12288

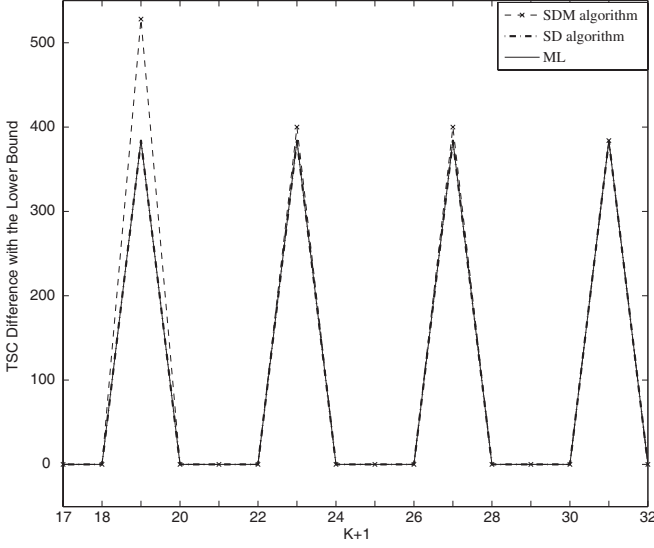


Fig. 1. TSC Difference with the lower bound by starting from optimal min-TSC set at all $K = 16, 17, \dots, 31$.

results to show the difference in TSC among SDM, SD, and ML methods. The most important contribution of the proposed signature assignment in this manuscript is that it always achieves the ML results.

In Fig. 2, we repeat the simulation in a different way. Instead of always starting from an optimal min-TSC signature set ($K, L = 16$), $K = 16, 17, \dots, 31$ and only adding one signature s_{K+1} by the proposed algorithm, we just initiate once from an optimal min-TSC signature set ($K = 16, L = 16$), and start upscaling signatures one-by-one consecutively by the proposed algorithm, all the way to a system of ($K + 1 = 32, L = 16$). In other words, the initial signature set at the intermediate steps is not necessarily min-TSC. As we can see from Fig. 2, the results of the proposed improved SD algorithm always reaches ML solution again, and outperform the previous suboptimal SDM algorithm, which demonstrates the effectiveness of our proposed algorithm for any initial signature set.

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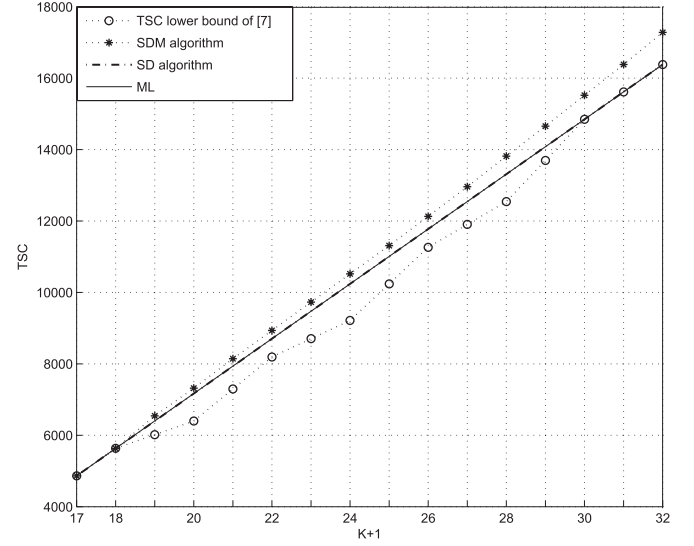


Fig. 2. TSC by starting from optimal min-TSC set only at $K = 16$.

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