$2^{\lfloor \log_2(M)+1 \rfloor}$. Then, we have counted the number of flops that a standard 1-D search algorithm require to locate the maxima abscissas and the number of flops required by Newton's method based on Horner's synthetic division [6, section 9.5]. They, respectively, required 3000 and 1788 real flops. (A real flop is the cost of computing a real sum or a real product.) These values are only approximate given that in the standard search algorithm, only the flops required to evaluate (8) were accounted for.

VIII. Conclusions

We have presented an efficient method to compute the Spectral and Root MUSIC estimations based on a conformal transformation. They can be calculated from the real roots of a real $(2M-2)$-degree polynomial that lie inside the $[-1, 1]$ range in Spectral MUSIC and from the complex conjugate roots of a real $(2M-2)$-degree polynomial in (Unitary) Root MUSIC. The calculation of the polynomial coefficients in both cases roughly requires $4M^2K$ real flops plus $2K$ times the computational cost of the convolution of two M-length vectors. For (Unitary) Root MUSIC, given that the resulting polynomial is real, the computational burden of the polynomial rooting step has been reduced.

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Estimate of Aliasing Error for Non-Smooth Signals Prefiltered by Quasi-Projections Into Shift-Invariant Spaces

Wen Chen, *Member, IEEE*, Bin Han, and Rong-Qing Jia

***Abstract*—An ideal analog-to-digital (A/D) converter prefilters a signal by an ideal lowpass filter. Recent research on A/D conversion based on shift-invariant spaces reveals that prefiltering signals by quasiprojections into shift-invariant spaces provides more flexible choices in designing an A/D conversion system of high accuracy. This paper focuses on the accuracy of such prefiltering, in which the aliasing error e_f^λ is found to behave like $\|e_f^\lambda\|_2 = O(\lambda^{-\alpha})$ with respect to the dilation λ of the underlying shift-invariant space, provided that the input signal f is Lipschitz-α continuous. A formula to calculate the coefficient of the decay rate is also figured out in this paper.**

***Index Terms*—A/D conversion, aliasing error, lowpass filter, prefiltering, quasiprojection, sampling, shift-invariant spaces, Strang–Fix condition, Wiener amalgam spaces.**

I. Introduction

In digital signal processing and digital communications, an analog signal is converted to a digital signal by an A/D (analog-to-digital) converter. An analog signal f is of *finite energy* if $\|f\|_2 < \infty$, where $\|f\|_2$ is the square norm of f defined by $\|f\|_2 = (\int_{\mathbb{R}} |f(t)|^2 dt)^{1/2}$. We also denote by $L^2(\mathbb{R})$ the signal space of finite energy, that is, $L^2(\mathbb{R}) = \{f : \|f\|_2 < \infty\}$. f is said to be *bandlimited* if $\hat{f}(\omega) = 0$ whenever $|\omega| > \sigma$ for some $\sigma > 0$, where $\hat{f}$ is the *Fourier transform* of f defined by $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-i\omega t}dt$. In this case, f is called a σ*-band signal*. An ideal A/D converter prefilters a signal of finite energy by an ideal lowpass filter (see Fig. 1). Then, the difference between the prefiltered signal and the original signal is referred to as the *aliasing error*. To reduce the aliasing error, one has to increase the bandwidth of the lowpass filter.

For a $\lambda \geq 1$, the *shift-invariant space* $V_\lambda(\varphi)$ generated by the *generator* $\varphi \in L^2(\mathbb{R})$ is defined as [3], [18]

$$V_\lambda(\varphi) = \left\{ \sum_{k\in\mathbb{Z}} c_k \varphi(\lambda \cdot -k) : \sum_{k\in\mathbb{Z}} |c_k|^2 < \infty \right\} \subset L^2(\mathbb{R}) \qquad (1)$$

where λ is called the *dilation* of the shift-invariant space $V_\lambda(\varphi)$. Let $\sin c\ t = \sin \pi t/\pi t$. Then, $V_\lambda(\sin c)$ is exactly the $\pi\lambda$-band signal space of finite energy, and hence, the ideal A/D conversion for a signal of finite energy is formulated as the A/D conversion based on $V_\lambda(\sin c)$ [2], [5], [7], [9], [10], [29], [30]. To prefilter an analog signal of finite energy by an ideal lowpass filter is then equivalent to making a *quasiprojection* $P^\lambda_{\sin c} : L^2(\mathbb{R}) \to V_\lambda(\sin c)$, that is, $P^\lambda_{\sin c}(f) = \lambda \sum_{k\in\mathbb{Z}} \langle f, \sin c(\lambda \cdot -k)\rangle \sin c(\lambda \cdot -k)$ for $f \in L^2(\mathbb{R})$, where $\langle \cdot, \cdot \rangle$ is the *inner product* in $L^2(\mathbb{R})$ defined by $\langle f, g\rangle = \int_{\mathbb{R}} f(t)g(t)dt$. Hence, the aliasing error is $e_f^\lambda = f - P^\lambda_{\sin c}(f)$, which can be made arbitrarily small by increasing the dilation λ of the shift-invariant space $V_\lambda(\sin c)$, i.e., the bandwidth of the ideal lowpass filter. This observation is very essential in the establishment of the *sampling theory in*

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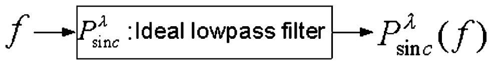

Fig. 1. Ideal A/D converter prefilters a signal f of finite energy by an ideal lowpass filter of bandwidth $\pi\lambda$. The output of the system is a $\pi\lambda$-band signal $P^{\lambda}_{\sin c}(f) \in V_{\lambda}(\sin c)$.

Fig. 2. Prefiltering a signal f of finite energy by a quasiprojection into shift-invariant space $V_{\lambda}(\varphi)$. The output of the system is $P^{\lambda}_{\varphi}(f) \in V_{\lambda}(\varphi)$. If $\varphi = \sin c$, it is the conventional prefiltering by an ideal lowpass filter of bandwidth $\pi\lambda$.

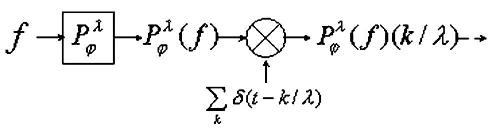

Fig. 3. Prefiltering and sampling: A signal f of finite energy is prefiltered by a quasi-interpolation P^{λ}_{φ} into a shift-invariant space $V_{\lambda}(\varphi)$. The prefiltered signal $P^{\lambda}_{\varphi}(f)$ is sampled by passing it through the unit pulse train $\sum_k \delta(\cdot - k/\lambda)$. The prefiltered signal $P^{\lambda}_{\varphi}(f)$ can be perfectly reconstructed from the samples $\{P^{\lambda}_{\varphi}(f)(k/\lambda)\}_k$ by Theorem 2. When $\varphi = \sin c$, this is the prefiltering by an ideal lowpass filter and sampling by Nyquist sampling theorem.

shift-invariant spaces [2], [5], [7], [9], [10], [29], [30], and the *prefiltering theory based on shift-invariant spaces* [4], [5], [28], which will be also addressed in this paper. Therefore, one can consider prefiltering a signal by *quasiprojections* into various shift-invariant spaces (see Figs. 2 and 3), that is, to project a signal of finite energy to a shift-invariant space by a quasiprojection $P^{\lambda}_{\varphi} : L^2(\mathbb{R}) \to V_{\lambda}(\varphi)$ defined by $P^{\lambda}_{\varphi}(f) = \lambda \sum_{k\in\mathbb{Z}} \langle f, \varphi(\lambda \cdot -k)\rangle \varphi(\lambda \cdot -k)$ for $f \in L^2(\mathbb{R})$.

In real-world applications, such an extension of prefiltering is useful and necessary, e.g., to perform nonideal prefiltering [25], to avoid the Gibbs phenomenon in the fast Fourier transform (FFT) [12], to use the impulse response of fast decay [23], to take into account real acquisition and reconstruction devices [29], to consider an arbitrary band signal [16], to obtain smoother frequency cutoff, or for numerical implementation [1], [2], [28], [32]. This is formulated by choosing an appropriate function φ with some desirable shape corresponding to a particular "impulse response" of a device, such as a compactly supported function, a function with polynomial or exponential decay, or a function φ with smooth cut-off frequency $\hat{\varphi}$. Then, one prefilters a signal by a quasiprojection into a *shift-invariant space* $V_{\lambda}(\varphi)$ and applying sampling theorem to the signals in $V_{\lambda}(\varphi)$ [2], [5], [7], [9], [10], [28], and [30].

Our objective in this paper is to estimate the aliasing error $e^{\lambda}_{f} = f - P^{\lambda}_{\varphi}(f)$ for a signal f of finite energy prefiltered by a quasiprojection into some shift-invariant space $V_{\lambda}(\varphi)$. We will prove that the aliasing error behaves like $\|e^{\lambda}_{f}\|_2 \le C_{\alpha,\varphi,f}\lambda^{-\alpha}$ with respect to the dilation λ of the underlying shift-invariant space $V_{\lambda}(\varphi)$, provided that φ satisfies the *Strang-Fix condition*, and f is Lipschitz-α continuous for some $\alpha \in (0, 1]$. Moreover, we will figure out a formula to calculate the coefficient $C_{\alpha,\varphi,f}$ of the decay rate, which is, however, unknown so far, even for smooth signals. We will also make a comparison with the conventional prefiltering by an ideal lowpass filter.

For a smooth signal, some kind of investigation has been done [4], [20], [21], [27]. In this paper, we focus, however, on the Lipschitz-α continuous signal space $\mathrm{Lip}^{\alpha}_{\mathrm{p}}$ for some positive number $\alpha \le 1$ and $p \ge 1$, which consists of all the measurable functions f for which the norm $\|f\|_{\mathrm{Lip}^{\alpha}_{\mathrm{p}}} = \sup_{s\in\mathbb{R}}(\|f - f(\cdot - s)\|_p/|s|^{\alpha}) < \infty$, where the norm $\|\cdot\|_p$ is defined by $\|f\|_p = (\int_{\mathbb{R}} |f(t)|^p dt)^{1/p}$. In a practical sense, this is an appropriate signal space since a practical signal is usually not smooth. Theoretically, however, a straightforward extension of our estimate can be applied to smooth signals as well.

II. Aliasing Error for Prefiltering by Quasi-Projections Into Shift-Invariant Spaces

In this section, we will estimate the aliasing error for prefiltering by a quasi-projection into shift-invariant space, which has been introduced in the introduction. Our analysis is performed in the the framework of *Wiener amalgam spaces* and the *Strang–Fix condition*. We will also briefly perform sampling in shift-invariant spaces and give a numerical result.

A. Wiener Amalgam Spaces and Weighted Wiener Amalgam Spaces

The *Wiener amalgam space* W, which is commonly used in sampling theory for shift-invariant spaces [5], [14], [15], consists of all measurable functions φ, for which the norm $\|\varphi\|_W = \sum_k \sup_{t\in[0,1]} |\varphi(t-k)| < \infty$. The *weighted Wiener amalgam space* W_r for $r > 0$ consists of all measurable functions φ, for which the norm $\|\varphi\|_{W_r} = \|(1+|\cdot|)^r\varphi\|_W < \infty$. In the remainder of this section, we assume that a continuous generator $\varphi \in W_1$, which means that the generator decays appropriately.

B. Strang–Fix Condition

We also need the *Strang–Fix condition*, which has been widely used in approximation by shift-invariant spaces [20], [21], [26]. A continuous generator $\varphi \in L^2(\mathbb{R})$ is said to satisfy the Strang–Fix condition if $\hat{\varphi}(2k\pi) = \delta(k)$ for $k \in \mathbb{Z}$, where δ is the Dirac sequence, which takes 1 at $k = 0$ and 0 at $k \ne 0$. One may think that the Strang–Fix condition is too strong. However, it is, in fact, a necessary condition for the aliasing error to decay with some order [20], [21]. By *Poisson summation formula* [7], one has $\sum_k \varphi(t-k) = \sum_k \hat{\varphi}(2k\pi)e^{-i2k\pi t}$. Therefore, the Strang–Fix condition is equivalent to $\sum_k \varphi(\cdot - k) = 1$. Obviously, any refinable function satisfies the *Strang–Fix condition* [11].

C. Prefiltering by Quasi-Projections Into Shift-invariant Spaces

For a continuous generator $\varphi \in W_1$ that satisfies the Strang–Fix condition, the prefiltering P^{λ}_{φ} by a quasi-projection into a shift-invariant space $V_{\lambda}(\varphi)$ has been defined in the introduction. Since $\varphi \in W_1$, one can extend P^{λ}_{φ} to the mapping $L^2(\mathbb{R}) \cup L^{\infty}(\mathbb{R}) \to V_{\lambda}(\varphi) \cup L^{\infty}(\mathbb{R})$. Since φ satisfies the Strang–Fix condition, for any constant $c \in \mathbb{R}$, we have $c = \sum_k c\varphi(\cdot - k)$, and hence, $\lambda \sum_k \langle c, \varphi(\lambda \cdot -k)\rangle \varphi(\lambda \cdot -k) = c\hat{\varphi}(0)\sum_k \varphi(\lambda \cdot -k) = c$. It shows that $P(c) = c$.

D. Aliasing Error for Prefiltering

In this subsection, we are going to estimate the aliasing error for a Lipschitz continuous signal prefiltered by a quasi-projection into a shift-invariant space. This is the main contribution of this paper. We at first derive an error estimate for a differentiable signal of finite energy. Then, we use it to obtain an error estimate for a Lipschitz continuous signal. Let $K(s,t) = \sum_k \varphi(s-k)\varphi(t-k)$. Since $\varphi \in W_1$, K is well defined. Then, we have the following estimate, the proof of which is presented in Appendix A.

Lemma 1: Suppose that a differentiable signal f of finite energy is prefiltered by a quasiprojection into a shift-invariant space $V_{\lambda}(\varphi)$. If

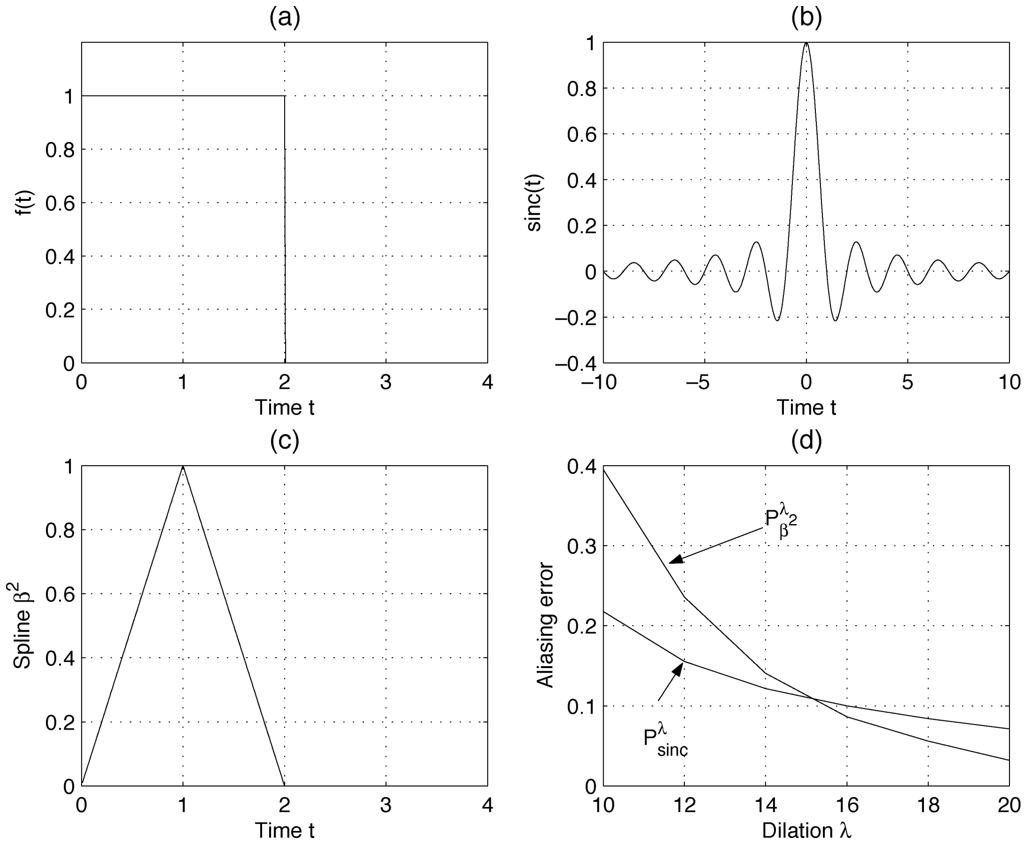

Fig. 4. (a) Input signal f, which is the characteristic function of the interval [0,2] and, hence, has singularities at $t = 0, 2$. (b) Sinc function, which is an everlasting function and decays by the rate $O(1/t)$. (c) B-spline of degree 2, β^2, which is supported on [0,2]. (d) Aliasing errors for prefiltering by a lowpass filter of bandwidth $\pi\lambda$ and by a quasiprojection into $V_\lambda(\beta^2)$ for dilation $\lambda = 10\ldots 20$.

the continuous generator $\varphi \in W_1$ satisfies the Strang–Fix condition, then the aliasing error behaves like

$$\left\|e_f^\lambda\right\|_2 \leq \lambda^{-3}\int_0^1 dt \left|\int_{\mathbb{R}} |(s-t)K(s,t)|\, ds \times \int_0^1 \left(\sum_\ell \left|f'\left(\frac{t}{\lambda}+\frac{\theta(s-t)}{\lambda}-\frac{\ell}{\lambda}\right)\right|^2\right)^{\frac{1}{2}} d\theta\right|^2.$$

∎

Now, we use the lemma to estimate the aliasing error for a Lipschitz continuous signal of finite energy. Let $L_\lambda = \lambda\cdot\chi_{[0,1]}(\lambda\cdot)$, where $\chi_{[0,1]}$ is the characteristic function of the closed interval [0,1] defined as $\chi_{[0,1]}(t)$, which takes 1 for $t \in [0,1]$ and takes 0 elsewhere. For a Lipschitz-α continuous signal $f \in L^2(\mathbb{R})$, we define the signal f_λ as $f_\lambda = f * L_\lambda * L_\lambda$, where $*$ is the convolution operator defined by $f*g = \int_{\mathbb{R}} f(s)g(\cdot - s)ds$. Since L_λ is supported on $[0, 1/\lambda]$, we have $f_\lambda(t) = \lambda\int_0^{1/\lambda} L_\lambda * f(t-s)ds = -\lambda\int_t^{t-1/\lambda} L_\lambda * f(y)dy$, which implies that f is differentiable and that $f'_\lambda = \lambda[L_\lambda * f - L_\lambda * f(\cdot - 1/\lambda)]$. Define the *autocorrelation filter* $G_\varphi = \sum_k |\hat{\varphi}(\cdot + 2k\pi)|^2$. Applying Lemma 1 to f_λ, we can derive the following estimate, the proof of which is presented in Appendix B.

Theorem 1: Suppose that a Lipschitz-α continuous signal f of finite energy is prefiltered by a quasiprojection into a shift-invariant space $V_\lambda(\varphi)$. If the continuous generator $\varphi \in W_1$ satisfies the Strang–Fix condition, then the aliasing error behaves like $\|e_f^\lambda\|_2 \leq C_{\alpha,\varphi,f}\lambda^{-\alpha}$, where the coefficient $C_{\alpha,\varphi,f} = (((1+\|G_\varphi\|_\infty)(2^{\alpha+2}-1)/(\alpha+1)(\alpha+2)) + \|\varphi\|_{W_1}^2)\|f\|_{\mathrm{Lip}_2^\alpha}$. ∎

Since $\alpha \in (0,1]$, it is easy to see that the decreasing function $(2^{\alpha+2}-1)/((\alpha+1)(\alpha+2)) \leq 3/2$. Therefore, the coefficient $C_{\alpha,\varphi,f}$ can be calculated by an inferior but simple formula $C^1_{\alpha,\varphi,f} = ((3/2)(1+\|G_\varphi\|_\infty) + \|\varphi\|^2_{W_1})\|f\|_{\mathrm{Lip}_2^\alpha}$. Consider a signal f that is prefiltered by a lowpass filter of bandwidth $\pi\lambda$. Let $I_\lambda = \chi_{[-\pi\lambda,\pi\lambda]}$. Then, the frequency response of the prefiltered signal is $\widehat{P^\lambda_{\mathrm{sinc}}(f)} = \hat{f}\chi_{I_\lambda}$. By the Parseval identity, the aliasing error is

$$\begin{aligned}\left\|e_f^\lambda\right\|_2 &= \left\|f - P^\lambda_{\mathrm{sinc}}(f)\right\|_2 = \frac{1}{\sqrt{2\pi}}\left\|\hat{f} - \widehat{P^\lambda_{\mathrm{sinc}}(f)}\right\|_2 \\ &= \frac{1}{2\pi}\left\|\hat{f} - \hat{f}\chi_{I_\lambda}\right\|_2 = \frac{1}{\sqrt{2\pi}}\left(\int_{|\omega|>\pi\lambda}\left|\hat{f}(\omega)\right|^2 d\omega\right)^{\frac{1}{2}}. \quad (2)\end{aligned}$$

If f is a Lipschitz-α continuous signal for some $\alpha \in (1/2, 1]$, one can obtain an estimate for (2) as $\|e_f^\lambda\|_2 \leq (\|f\|_{\mathrm{Lip}_1^\alpha}/2\sqrt{2\alpha-1})\lambda^{-\alpha+1/2}$ (see Appendix C). Then, aliasing error for the ideal lowpass prefiltering decays by the order $O(\lambda^{-\alpha+1/2})$, which is obviously slower than the order $O(\lambda^{-\alpha})$, which is the decay rate of aliasing error for prefiltering by a quasi-projection into an appropriate shift-invariant space.

E. Sampling in Shift-invariant Spaces

An A/D conversion includes the consecutive processes of prefiltering, sampling, and quantization. Since quantization is beyond the scope of this paper, the discussion will be omitted (see Fig. 4). See our recent manuscript [5] for details. For a signal prefiltered by an ideal lowpass filter, one can apply *Nyquist sampling theorem* to conduct sampling. Then, the prefiltered signal $P^\lambda_{\mathrm{sinc}}(f)$ can be perfectly reconstructed from the samples $\{P^\lambda_{\mathrm{sinc}}(f)(k/\lambda)\}_k$

by $P^{\lambda}_{\sin \mathrm{c}}(f) = \sum_k P^{\lambda}_{\sin \mathrm{c}}(f)(k/\lambda) \sin \mathrm{c}(\lambda \cdot -k)$. Similarly, one can extend Nyquist sampling theorem to the signals in shift-invariant spaces [2], [7], [8], [31], [32]. Denote the discrete time Fourier transfrom (DTFT) of the samples $\{\varphi(k)\}_k$ of φ by $\hat{\varphi}^*(\omega) = \sum_k \varphi(k)e^{-ik\omega}$. It has been shown that there is an *admissible reconstruction filter* $S_\varphi \in W_1$ determined by $\hat{S}_\varphi = \hat{\varphi}/\hat{\varphi}^*$, such that $g = \sum_k g(k)S_\varphi(\cdot - k)$ for $g \in V_1(\varphi)$, provided that $\hat{\varphi}^* \neq 0$. For any $g \in V_\lambda(\varphi)$, we have $g(\cdot/\lambda) \in V_1(\varphi)$. Therefore, $g(\cdot/\lambda) = \sum_k g(k/\lambda)S_\varphi(\cdot - k)$. Consequently, $g = \sum_k g(k/\lambda)S_\varphi(\lambda \cdot -k)$. Applying this result to the prefiltered signal $P^{\lambda}_{\varphi}(f)$, we have the sampling theorem for the shift-invariant space $V_\lambda(\varphi)$ as follows.

Theorem 2: Suppose that a signal f of finite energy is prefiltered by a quasi-projection P^{λ}_{φ} into a shift-invariant space $V_\lambda(\varphi)$. If the continuous generator $\varphi \in W_1$ such that $\hat{\varphi}^* \neq 0$, then the prefiltered signal can be perfectly reconstructed by $P^{\lambda}_{\varphi}(f) = \sum_k P^{\lambda}_{\varphi}(f)(k/\lambda)S_\varphi(\lambda \cdot -k)$. ■

Since $\widehat{\sin \mathrm{c}}^* = 1$, this result coincides with the conventional Nyquist sampling theorem. Sampling theorem for shift-invariant spaces has been also extended to the nonuniform sampling in various cases. See the investigations in [1], [5], [6], [9], [10], [22], and [28]. Since sinc function slowly decays as time goes to infinity, the conventional reconstruction is very sensitive to noise. However, one can choose a generator φ such that $\hat{S}_\varphi$ decays rapidly. In some extreme cases, one can design a compactly supported $\hat{S}_\varphi$ (see the next subsection). Therefore, reconstruction by Theorem 2 will not be sensitive to noise and converge rapidly, which will meanwhile reduce the computational complexity. This is another advantage of prefiltering a signal by a quasi-projection into a shift-invariant space.

F. Numerical Results

We give a numerical example to demonstrate the prefiltering and sampling based on the *B-spline* shift-invariant space. A B-spline of degree N is defined by $\beta^N = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$, which is the N-times convolution of the characteristic function $\chi_{[0,1]}$. Then, $\widehat{\beta^N} = (1 - e^{-i\omega}/i\omega)^N$, which obviously satisfies the Strang–Fix condition. Suppose that a Lipschitz-α continuous signal f is prefiltered by a quasi-projection $P^{\lambda}_{\beta^N}$ into the shift-invariant space $V_\lambda(\beta^N)$. By Theorem 1, the aliasing error satisfies $\|e^{\lambda}_{f}\|_2 \leq C_{\alpha,\beta^N,f}\lambda^{-\alpha}$, where $C_{\alpha,\beta^N,f} = (((1+\|G_{\beta^N}\|_\infty)(2^{\alpha+2}-1)/((\alpha+1)(\alpha+2))) + \|\beta^N\|^2_{W_1})\|f\|_{\mathrm{Lip}^{\alpha}_2}$. Consider the input signal $f(t) = \chi_{[0,2]}$. Obviously, f has singularities at 0 and 2 [see Fig. 4(a)]. Since $f \in \mathrm{Lip}^{0.5}_2$, it is deduced that $\|e^{\lambda}_{f}\|_2 \leq O(\lambda^{-0.5})$. For simplicity, we consider the B-spline of degree 2, β^2, the graph of which is shown in Fig. 4(c). By Matlab, we find $\|G_{\beta^2}\|_\infty = 3$ and $\|\varphi\|_{W_1} = 2$. Then, the aliasing error satisfies $\|e^{\lambda}_{f}\|_2 \leq 6\lambda^{-0.5}$. Fig. 4(d) shows the actual aliasing error for $\lambda = 10, \ldots, 20$. The actual aliasing error is lower than the estimate, which implies that our estimate is not optimal. We also put the aliasing error for the ideal lowpass prefiltering in Fig. 4(d). Since $f \in \mathrm{Lip}^1_1$, the theoretical estimate of aliasing error for ideal lowpass prefiltering also behaves like $O(\lambda^{-0.5})$, but Fig. 4(d) tells us that the actual accuracy of prefiltering $P^{\lambda}_{\beta^2}$ is superior to that of $P^{\lambda}_{\sin \mathrm{c}}$, even if their theoretical estimates are in the same order. Visually, we will see that prefiltering by a quasi-projection into $V_\lambda(\beta^2)$ provides good approximation. In Fig. 5, the ideal lowpass prefiltering $P^{\lambda}_{\sin \mathrm{c}}$ introduces small ripples in the smooth part of f and big ripples at the singularities $t = 0$, 2. However, $P^{\lambda}_{\beta^2}(f)$ approximates f very well. For this special signal, $f \in \mathrm{Lip}^{0.5}_2 \cap \mathrm{Lip}^1_1$. However, in most cases, $f \in \mathrm{Lip}^{\alpha}_2 \cap \mathrm{Lip}^{\alpha}_1$ for some $\alpha \in (0, 1]$, e.g., $f \in \mathrm{Lip}^1_1 \cap \mathrm{Lip}^1_2$ if f is smooth. Hence, prefiltering by quasi-projection into shift-invariant space provides higher accuracy.

One the other hand, since $\hat{\beta^2}^*(\omega) = e^{-i\omega}$, by Theorem 2, $S_{\beta^2} = \beta^2(\cdot + 1)$. Since β^2 is compactly supported, computing for finite terms by Theorem 2 will perfectly reconstruct $P^{\lambda}_{\beta^2}(f)$ for any signal f of finite energy. Since sinc function is an everlasting function [see Fig. 4(b)], one has to compute for infinite terms to perfectly reconstruct $P^{\lambda}_{\sin \mathrm{c}}(f)$. This reduces the computational complexity in reconstruction.

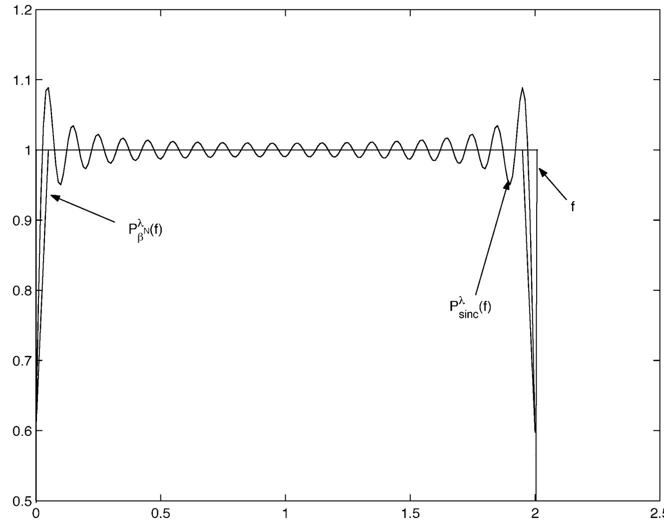

Fig. 5. Ideal lowpass prefiltering $P^{\lambda}_{\sin \mathrm{c}}$ and the prefiltering $P^{\lambda}_{\beta^2}$ by a quasi-projection into $V_\lambda(\beta^2)$ for $\lambda = 20$. This graph is based on the same computational complexity. Visually, $P^{\lambda}_{\beta^2}(f)$ provides better approximation to f than $P^{\lambda}_{\sin \mathrm{c}}$ does, since $P^{\lambda}_{\sin \mathrm{c}}(f)$ introduces small ripples in the smooth part of f and big ripples at the singularities of f.

III. Conclusion

In this paper, we introduce a novel method of prefiltering a signal of finite energy by a quasi-projection into some shift-invariant spaces. In such a prefiltering, we find that the aliasing error behaves like $O(\lambda^{-\alpha})$ if the input signal is Lipschitz-α continuous. An explicit formula to calculate the coefficient of the decay rate is figured out. Meanwhile, we make comparison with the ideal lowpass prefiltering theoretically and numerically. Therefore, it provides various choices for one to design an A/D conversion system of high accuracy, low computational complexity (by choosing some compactly supported φ), efficient reconstruction for sampling (by choosing some φ such that S_φ is compactly supported or of fast decay). Future applications include being suitable for nonideal prefiltering (by choosing some φ such that $\hat{\varphi}$ has smooth cut off); being suitable for an arbitrary band signal (by choosing some φ such that $\hat{\varphi}$ matches the practical bands); and designing an optimal prefiltering (by choosing some φ such that $\|G_\varphi\|_\infty$ and $\|\varphi\|_{W_1}$ are both small).

Appendix A
Proof of Lemma 1

By *Taylor's theorem*, we have $f(s) = f(t) + \int_0^1 f'(t + \theta(s - t))(s - t)d\theta$. For any $\ell \in \mathbb{Z}$, by the definition of prefiltering, we have $P^{\lambda}_{\varphi}(f)(t - \ell/\lambda) = \lambda \sum_k \langle f, \varphi(\lambda \cdot -k)\rangle \varphi(\lambda(t - \ell/\lambda) - k) = \lambda \sum_k \langle f, \varphi(\lambda \cdot + \ell - k)\rangle \varphi(\lambda t - k) = \lambda \sum_k \langle f(\cdot - \ell/\lambda), \varphi(\lambda \cdot -k)\rangle \varphi(\lambda t - k)$. Notice $f(t - \ell/\lambda) = \lambda \sum_k \langle f(t - \ell/\lambda), \varphi(\lambda \cdot -k)\rangle \varphi(\lambda t - k)$. We have

$$P^{\lambda}_{\varphi}(f)\left(t - \frac{\ell}{\lambda}\right) - f\left(t - \frac{\ell}{\lambda}\right) = \lambda \sum_k \varphi(\lambda t - k) \int_{\mathbb{R}} \left[f\left(s - \frac{\ell}{\lambda}\right) - f\left(t - \frac{\ell}{\lambda}\right)\right] \times \varphi(\lambda s - k)ds$$

$$
\begin{aligned}
&= \lambda \sum_k \varphi(\lambda t - k) \int_{\mathbb{R}} \varphi(\lambda s - k) ds \\
&\quad \cdot \int_0^1 f'\left[t - \frac{\ell}{\lambda} + \theta(s-t)\right](s-t) d\theta \\
&= \sum_k \varphi(\lambda t - k) \int_{\mathbb{R}} \varphi(s-k) ds \\
&\quad \cdot \int_0^1 f'\left[t - \frac{\ell}{\lambda} + \theta\left(\frac{s}{\lambda} - t\right)\right]\left(\frac{s}{\lambda} - t\right) d\theta.
\end{aligned}
$$

Let $x = t/\lambda + \theta(s-t)/\lambda$. Then

$$
\begin{aligned}
&\left\| P_\varphi^\lambda(f) - f \right\|_2^2 \\
&= \int_0^{\frac{1}{\lambda}} \sum_\ell \left| P_\varphi^\lambda(f)\left(t - \frac{\ell}{\lambda}\right) - f\left(t - \frac{\ell}{\lambda}\right)\right|^2 dt \\
&= \lambda^{-1} \int_0^1 \sum_\ell \left| P_\varphi^\lambda(f)\left(\frac{t}{\lambda} - \frac{\ell}{\lambda}\right) - f\left(\frac{t}{\lambda} - \frac{\ell}{\lambda}\right)\right|^2 dt \\
&= \lambda^{-3} \int_0^1 dt \sum_\ell \left| \int_{\mathbb{R}} (s-t)K(s,t) ds \int_0^1 f'\left(x - \frac{\ell}{\lambda}\right) d\theta \right|^2 \\
&\leq \lambda^{-3} \int_0^1 dt \left| \int_{\mathbb{R}} |(s-t)K(s,t)| \, ds \cdot \int_0^1 d\theta \right. \\
&\qquad \left. \times \left(\sum_\ell \left| f'\left(\frac{t}{\lambda} + \frac{\theta(s-t)}{\lambda} - \frac{\ell}{\lambda}\right)\right|^2 \right)^{\frac{1}{2}} \right|^2.
\end{aligned}
$$

Appendix B
Proof of Theorem 1

Since $f'_\lambda = \lambda[L_\lambda * f - L_\lambda * f(\cdot - 1/\lambda)]$, we have

$$
\begin{aligned}
\left| f'_\lambda(t) \right|^2 &= \left| \lambda \int_0^{\frac{1}{\lambda}} \left[f(t-s) - f\left(t - \frac{1}{\lambda} - s\right)\right] L_\lambda(s) ds \right|^2 \\
&\leq \lambda^2 \int_0^{\frac{1}{\lambda}} \left| f(t-s) - f\left(t - \frac{1}{\lambda} - s\right)\right|^2 ds \cdot \int_0^{\frac{1}{\lambda}} |L_\lambda(s)|^2 ds \\
&= \lambda^3 \int_0^{\frac{1}{\lambda}} \left| f(t-s) - f\left(t - s - \frac{1}{\lambda}\right)\right|^2 ds \cdot \int_0^1 \chi_{[0,1]} ds \\
&= \lambda^3 \int_0^{\frac{1}{\lambda}} \left| f(t-s) - f\left(t - s - \frac{1}{\lambda}\right)\right|^2 ds.
\end{aligned}
$$

Since $f \in \mathrm{Lip}_2^\alpha$, it shows that

$$
\begin{aligned}
&\sum_\ell \left| f'_\lambda\left(t + \frac{\ell}{\lambda}\right)\right|^2 \\
&\leq \lambda^3 \sum_\ell \int_0^{\frac{\ell}{\lambda}} \left| f\left(t + \frac{\ell}{\lambda} - s\right) - f\left(t + \frac{\ell}{\lambda} - s - \frac{1}{\lambda}\right)\right|^2 \qquad (3) \\
&= \lambda^3 \sum_\ell \int_{\frac{(\ell-1)}{\lambda}}^{\frac{\ell}{\lambda}} \left| f(t-s) - f\left(t - s - \frac{1}{\lambda}\right)\right|^2 ds \\
&= \lambda^3 \left\| f(t - \cdot) - f\left(t - \frac{1}{\lambda} - \cdot\right)\right\|_2^2 \\
&\leq \lambda^{3-2\alpha} \|f\|^2_{\mathrm{Lip}_2^\alpha}. \qquad (4)
\end{aligned}
$$

By Lemma 1 and (4), we have $\|f_\lambda - P_\varphi^\lambda(f_\lambda)\|_2^2 \leq \lambda^{-2\alpha}\|f\|^2_{\mathrm{Lip}^\alpha} \sup_{s\in\mathbb{R}}(\int_{\mathbb{R}} |(s-t)K(s,t)| dt)^2$. On the other hand, we also have

$$
\begin{aligned}
&\int_{\mathbb{R}} |(s-t)K(s,t)| \, dt \\
&= \sum_k \int_{\mathbb{R}} |(s-t-k+k)\varphi(s-k)\varphi(t-k)| \, dt \\
&\leq \sum_k \int_{\mathbb{R}} (1+|s-k|)(1+|t-k|) |\varphi(s-k)\varphi(t-k)| \, dt \\
&\leq \|\varphi\|^2_{W_1}. \qquad (5)
\end{aligned}
$$

It is deduced that $\|f_\lambda - P_\varphi^\lambda(f_\lambda)\|_2 \leq \|f\|_{\mathrm{Lip}_2^\alpha}\|\varphi\|^2_{W_1}\lambda^{-\alpha}$. By the definition of L_λ, we have $L_\lambda * L_\lambda = \int_0^{1/\lambda} \lambda L_\lambda(t-s) ds = \lambda \int_0^1 \chi_{[0,1]}(\lambda t - s) ds$, which implies that $\int_{\mathbb{R}} L_\lambda * L_\lambda(t) dt = \int_0^1 ds \int_{\mathbb{R}} \lambda L(\lambda t - s) dt = \int_0^1 ds \int_{\mathbb{R}} \chi_{[0,1]}(t) dt = 1$. Since $f \in \mathrm{Lip}_2^\alpha$, we have

$$
\begin{aligned}
\|f_\lambda - f\|_2 &= \left\| \int_{\mathbb{R}} [f(\cdot - s) - f(\cdot)] (L_\lambda * L_\lambda)(t) dt \right\|_2 \\
&= \int_{\mathbb{R}} \|f(\cdot - t) - f(\cdot)\|_2 \, |(L_\lambda * L_\lambda)(t)| \, dt \\
&\leq \|f\|_{\mathrm{Lip}_2^\alpha} \int_{\mathbb{R}} |t|^\alpha \, |(L_\lambda * L_\lambda)(t)| \, dt \\
&= \|f\|_{\mathrm{Lip}_2^\alpha} \int_{\mathbb{R}} |t|^\alpha dt \int_0^1 \lambda\chi|_{[0,1]}(\lambda t - s) ds \\
&= \lambda^{-\alpha} \|f\|_{\mathrm{Lip}_2^\alpha} \int_{\mathbb{R}} |t|^\alpha dt \int_0^1 \chi_{[0,1]}(t-s) ds. \qquad (6)
\end{aligned}
$$

Since $\int_0^1 \chi_{[0,1]}(t-s)ds = t\chi_{[0,1]} + (2-t)\chi_{[1,2]}$, we have $\|f_\lambda - f\|_2 \leq ((2^{\alpha+2} - 1/((\alpha+1)(\alpha+2)))\lambda^{-\alpha}\|f\|_{\mathrm{Lip}_2^\alpha}$. Finally, the aliasing error is

$$
\begin{aligned}
\left\| e_f^\lambda \right\|_2 &\leq \|f - f_\lambda\|_2 + \left\| f_\lambda - P_\varphi^\lambda(f_\lambda)\right\|_2 + \left\| P_\varphi^\lambda(f_\lambda) - P_\varphi^\lambda(f)\right\|_2 \\
&\leq \lambda^{-\alpha}\|f\|_{\mathrm{Lip}^\alpha} \left(\frac{(1 + \|P_\varphi^\lambda\|_2)(2^{\alpha+2} - 1)}{(\alpha+1)(\alpha+2)} + \|\varphi\|^2_{W_1} \right)
\end{aligned}
$$

where $\|P_\varphi^\lambda\|_2$ is the norm of the quasi-projection P_φ^λ defined by $\|P_\varphi^\lambda\|_2 = \sup_{f \in L^2(\mathbb{R})}(\|P_\varphi^\lambda(f)\|_2/\|f\|_2)$. In order to prove Theorem

1, we need to prove $\|P_\varphi^\lambda\|_2 \le \|G_\varphi\|_\infty$ only. By the Parseval identity, we have

$$\begin{aligned}
&\left\|P_\varphi^\lambda(f)\right\|_2 \\
&= \left\|\lambda\sum_k \langle f, \varphi(\lambda\cdot -k)\rangle \varphi(\lambda\cdot -k)\right\|_2 \\
&= \frac{1}{\sqrt{2\pi}}\left\|\hat\varphi\left(\frac{\cdot}{\lambda}\right)\sum_k \frac{1}{2\pi\lambda}\left\langle \hat f\bar{\hat\varphi}\left(\frac{\cdot}{\lambda}\right), e^{\frac{ki\cdot}{\lambda}}\right\rangle e^{-\frac{ik\cdot}{\lambda}}\right\|_2 \\
&= \frac{1}{\sqrt{2\pi}}\left(\int_0^{2\pi\lambda} G_\varphi\left(\frac{\omega}{\lambda}\right)\right. \\
&\qquad \left.\cdot\left|\sum_k \hat f(\omega+2k\pi\lambda)\bar{\hat\varphi}\left(\frac{\omega}{\lambda}+2k\pi\right)\right|^2 d\omega\right)^{\frac12} \\
&\le \frac{1}{\sqrt{2\pi}}\sqrt{\|G_\varphi\|_\infty}\left(\int_0^{2\pi}\sum_k\left|\hat f(\omega+2k\pi)\right|^2\right. \\
&\qquad \left.\cdot\sum_k\left|\hat\varphi\left(\frac{\omega}{\lambda}+2k\pi\right)\right|^2 d\omega\right)^{\frac12} \\
&\le \frac{1}{\sqrt{2\pi}}\|G_\varphi\|_\infty\|\hat f\|_2 \\
&= \|G_\varphi\|_\infty\|f\|_2.
\end{aligned}$$

Therefore, $\|P_\varphi^\lambda\|_2 \le \|G_\varphi\|_\infty$. This completes the proof.

Appendix C
Aliasing Error for Prefiltering by an Ideal Lowpass Filter of Bandwidth $\pi\lambda$

By the definition of the Fourier transform, for $\omega \ne 0$, we have

$$\begin{aligned}
\hat f(\omega) &= \int_{\mathbb{R}} f(t)e^{-it\omega}dt \\
&= \int_{\mathbb{R}} f\left(t-\frac{\pi}{\omega}\right)e^{-i\left(t-\frac{\pi}{\omega}\right)\omega}dt \\
&= -\int_{\mathbb{R}} f\left(t-\frac{\pi}{\omega}\right)e^{-it\omega}dt.
\end{aligned}$$

Since $f \in \mathrm{Lip}_1^\alpha$, we have

$$\begin{aligned}
\left|2\hat f(\omega)\right| &= \left|\int_{\mathbb{R}}\left(f(t)-f\left(t-\frac{\pi}{\omega}\right)\right)e^{-i\omega t}dt\right| \\
&\le \left\|f-f\left(\cdot-\frac{\pi}{\omega}\right)\right\|_1 \\
&\le \|f\|_{\mathrm{Lip}_1^\alpha}\left|\frac{\pi}{\omega}\right|^\alpha.
\end{aligned}$$

Therefore, the aliasing error satisfies

$$\begin{aligned}
\left\|e_f^\lambda\right\|_2^2 &\le \frac{\|f\|^2_{\mathrm{Lip}_1^\alpha}\pi^{2\alpha}}{8\pi}\int_{|\omega|>\pi\lambda}|\omega|^{-2\alpha}d\omega \\
&= \frac{\|f\|^2_{\mathrm{Lip}_1^\alpha}\pi^{2\alpha}}{4\pi(2\alpha-1)}(\pi\lambda)^{-(2\alpha-1)} \\
&= \frac{\|f\|^2_{\mathrm{Lip}_1^\alpha}}{4(2\alpha-1)}\lambda^{-(2\alpha-1)}
\end{aligned}$$

that is $\|e_f^\lambda\|_2 \le (\|f\|_{\mathrm{Lip}_1^\alpha}/2\sqrt{2\alpha-1})\lambda^{-(\alpha-1/2)}$.

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Robust Super-Exponential Methods for Deflationary Blind Source Separation of Instantaneous Mixtures

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***Abstract*—The so-called “super-exponential” methods (SEMs) are attractive methods for solving blind signal processing problems. The conventional SEMs, however, have such a drawback that they are very sensitive to Gaussian noise. To overcome this drawback, we propose a new SEM. While the conventional SEMs use the second- and higher order cumulants of observations, the proposed SEM uses only the higher order cumulants of observations. Since higher order cumulants are insensitive to Gaussian noise, the proposed SEM is robust to Gaussian noise, which is referred to as a robust super-exponential method (RSEM). To show the validity of the proposed RSEM, some simulation results are presented.**

***Index Terms*—Blind source separation, deflationary approach, Gaussian noise, instantaneous mixtures, super-exponential methods.**

I. Introduction

This correspondence deals with the blind source separation (BSS) problem of a static system driven by (or linear mixtures of) independent source signals. To solve this problem, the ideas of the super-exponential methods (SEMs) in [1], [4], and [6] are used. Several researchers (e.g., [1], [4]–[6], [10]) have so far proposed some SEMs for solving independent component analysis (ICA), blind deconvolution (BD), and blind channel equalization (BCE). One of the attractive properties of the SEMs is that they are computationally efficient and that they converge to a desired solution at a super-exponential rate. However, almost all the conventional SEMs have the drawback that they are very sensitive to Gaussian noise (this will be shown in Section IV) because they utilize the second- and higher order cumulants of observations.

In this correspondence, we propose a new SEM that overcomes the drawback. The proposed SEM utilizes only the higher order cumulants of observations, and hence, the proposed SEM becomes robust to Gaussian noise, which is referred to as a *robust super-exponential method* (RSEM). Simulation results show that the proposed RSEM is robust to Gaussian noise and can successfully achieve the BSS of static systems (or linear mixtures of independent source signals).

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II. Problem Formulation

Throughout this correspondence, let us consider the following MIMO static system with n inputs and m outputs:

$$\boldsymbol{y}(t) = \boldsymbol{H}\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{1}$$

where $\boldsymbol{y}(t)$ represents an m-column output vector called the *observed signal*, $\boldsymbol{s}(t)$ represents an n-column input vector called the *source signal*, $\boldsymbol{H}$ is an $m \times n$ matrix, and $\boldsymbol{n}(t)$ represents an m-column noise vector. It can be regarded as a linear mixture model with additive noise.

To achieve the blind source separation (BSS) for the system (1), the following n filters, which are m-input single-output (MISO) static systems driven by the observed signals, are used:

$$z_l(t) = \boldsymbol{w}_l^T \boldsymbol{y}(t), \quad l = 1, 2, \cdots, n \tag{2}$$

where $z_l(t)$ is the lth output of the filter, and $\boldsymbol{w}_l = [w_{l1}, w_{l2}, \cdots, w_{lm}]^T$ is an m-column vector representing the m coefficients of the filter. Substituting (1) into (2), we obtain

$$\begin{aligned} z_l(t) &= \boldsymbol{w}_l^T \boldsymbol{H}\boldsymbol{s}(t) + \boldsymbol{w}_l^T \boldsymbol{n}(t) \\ &= \boldsymbol{g}_l^T \boldsymbol{s}(t) + \boldsymbol{w}_l^T \boldsymbol{n}(t), \quad l = 1, 2, \cdots, n \end{aligned} \tag{3}$$

where $\boldsymbol{g}_l = [g_{l1}, g_{l2}, \cdots, g_{ln}]^T := \boldsymbol{H}^T \boldsymbol{w}_l$ is an n-column vector. The BSS problem considered in this correspondence can be formulated as follows: Find n filters $\boldsymbol{w}_l$'s denoted by $\tilde{\boldsymbol{w}}_l$'s satisfying the following condition, without the knowledge of $\boldsymbol{H}$, even if the Gaussian noise $\boldsymbol{n}(t)$ is added to the observed signal $\boldsymbol{y}(t)$

$$\tilde{\boldsymbol{g}}_l = \boldsymbol{H}^T \tilde{\boldsymbol{w}}_l = \tilde{\boldsymbol{\delta}}_l, \quad l = 1, 2, \cdots, n \tag{4}$$

where $\tilde{\boldsymbol{\delta}}_l$ is an n-column vector whose elements $\tilde{\delta}_{lr}$ $(r = 1, 2, \cdots, n)$ are equal to zero, expect for the ρ_lth element, that is, $\tilde{\delta}_{lr} = c_l \delta(r - \rho_l)$, $r = 1, 2, \cdots, n$.

Here, $\delta(t)$ is the Kronecker delta function, c_l is a number standing for a scale change, and ρ_l is one of integers $\{1, 2, \cdots, n\}$ such that the set $\{\rho_1, \rho_2, \cdots, \rho_n\}$ is a permutation of the set $\{1, 2, \cdots, n\}$.

To solve the BSS problem, we put the following assumptions on the system and the source signals.

A1) The matrix $\boldsymbol{H}$ in (1) is an $m \times n$ $(m \geq n)$ matrix and has full column rank.

A2) The input sequence $\{\boldsymbol{s}(t)\}$ is a zero-mean, non-Gaussian vector stationary process whose element processes $\{s_i(t)\}$, $i = 1, 2, \cdots, n$ are mutually statistically independent and have nonzero $(p+1)$st-order cumulants κ_i defined as

$$\kappa_i = \mathrm{cum}\{\underbrace{s_i(t), s_i(t), \cdots, s_i(t)}_{p+1}\} \neq 0 \tag{5}$$

where $i = 1, 2, \cdots, n$, and $p \geq 2$.