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On Sampling in Shift Invariant Spaces

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***Abstract*—In this correspondence, a necessary and sufficient condition for sampling in the general framework of shift-invariant spaces is derived. Then this result is applied, respectively, to the regular sampling and the perturbation of regular sampling in shift-invariant spaces. A simple necessary and sufficient condition for regular sampling in shift-invariant spaces is attained. Furthermore, an improved estimate for the perturbation is derived for the perturbation of regular sampling in shift-invariant spaces. The derived estimate is easy to calculate, and shown to be optimal in some shift-invariant spaces. The algorithm to calculate the reconstruction frame is also presented in this correspondence.**

***Index Terms*—Frame, generator, irregular sampling, sampling, shift-invariant space, Zak transform.**

I. Introduction and Preliminaries

In digital signal and image processing and digital communications, a continuous signal is usually represented and processed by using its discrete samples. Then a fundamental problem is how to represent a continuous signal in terms of a discrete sequence. For a band-limited signal of finite energy, it is completely characterized by its samples, by the famous classical *Shannon sampling theorem*. Observing that the Shannon function $\mathrm{sinc} = \frac{\sin \pi(\cdot)}{\pi(\cdot)}$ is, in fact, a *scaling function* of a *multiresolution analysis* (MRA) [11], Walter [19] first extended Shannon sampling theorem to the setting of *wavelet subspaces*. Following Walter's work [19], Janssen [13] studied the so-called *shift sampling* in wavelet subspaces by using the *Zak transform*, which can cover some extra cases of Walter's result. Walter [20], Xia [22], and the authors of [7], [8] also studied the *oversampling* procedure in wavelet subspaces. On the other hand, Aldroubi and Unser [1]–[3], [18] studied the sampling procedure in *shift-invariant spaces*. Chen and Itoh [9] improved the works

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of Walter [19], and Aldroubi and Unser [3], and found a necessary and sufficient condition for sampling in shift-invariant spaces.

However, in many real applications, sampling points are not always measured regularly. Sometimes the sampling steps need to be fluctuated according to the signals so as to reduce the number of samples and computational complexity. There are also many cases where undesirable jitter exists in sampling instants. Some communication systems may suffer from random delay due to channel traffic congestion and encoding delay. In such cases, the system will become more efficient when a perturbation factor is considered. For the band-limited signals of finite energy, a generalization of the Shannon sampling theorem, known as *Kadec's theorem* [24], can be used. Following the work on sampling in wavelet subspaces, Liu and Walter [16], Liu [15], and the authors [4] tried to extend Kadec's theorem to a class of wavelet subspaces. But they actually did not get a real extension of Kadec's theorem. Then the authors [6] introduced a function class $L_\sigma^\lambda[a, b]$ ($\lambda > 0$, $\sigma \in [0, 1)$, and $0 \in [a, b] \subset [-1, 1]$), a norm $\|\cdot\|_{L_\sigma^\lambda[a, b]}$ of $L_\sigma^\lambda[a, b]$, and found an algorithm to treat the perturbation of regular sampling in wavelet subspaces as follows. Some notations used in the theorem will be defined in the next sections.

Theorem 1 [6]: Suppose that φ is a continuous scaling function of an MRA $\{V_m\}_m$ in $L_\sigma^\lambda[a, b]$. If $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1$, and the Zak transform $Z_\varphi(\sigma, \cdot) \neq 0$, then there is an $r_{\sigma,\varphi} \in (0, 1]$ such that for any scalar sequence $\{r_k\}_k \subset [-r_{\sigma,\varphi}, r_{\sigma,\varphi}] \cap [a, b]$, there is a Riesz basis[1] $\{S_{\sigma,k}\}_k$ of V_0 such that

$$f = \sum_k f(k + \sigma + r_k) S_{\sigma,k} \tag{1}$$

holds in $L^2(R)$ for any $f \in V_0$ if

$$r_{\sigma,\varphi} < \left(\frac{\|Z_\varphi(\sigma,\cdot)G_\varphi\|_0 \|Z_\varphi(\sigma,\cdot)/G_\varphi\|_0}{\|q_\varphi(\cdot,\sigma)\|_{L_\sigma^\lambda[a,b]}} \right)^{1/\lambda}. \tag{2}$$

Apply the theorem to calculating the *B-spline of degree* 1

$$N_1(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)}$$

where $\chi_{[0,1)}$ is the characteristic function of the interval $[0, 1)$, so is $\chi_{[1,2)}$. From (2), we get the estimate $r_{0,N_1} < 1/(2\sqrt{3})$ when $r_k \geq 0$ (or $r_k \leq 0$) for all $k \in Z$. But Liu and Walter [16] showed that the estimate $r_{0,N_1} < 1/2$ is optimal for B-spline of degree 1. This implies that the authors' result [6] is not optimal.

Our objective in this correspondence is to find a necessary and sufficient condition for general nonuniform sampling in shift-invariant spaces by using the *frame theory*. By applying the results, respectively, to regular sampling and perturbation of regular sampling in shift-invariant spaces, we try to find the optimal criterion for conducting regular sampling and perturbation of regular sampling in shift-invariant spaces. Fortunately, for regular sampling in shift-invariant spaces, a simple necessary and sufficient condition is found; for perturbation of regular sampling, improved estimates for the perturbation are derived. By applying the new estimate to the B-spline of degree 1, we derive the optimal estimate $r_{0,N_1} < 1/2$ when $r_k \geq 0$ (or $r_k \leq 0$) for all $k \in Z$.

[1]A sequence of vectors $\{x_k\}$ in an infinite-dimensional Banach space X is said to be a *Schauder basis* for X if to each vector x in the space corresponds a unique sequence of scalars $\{c_k\}_k$ such that $x = \sum_{k=-\infty}^{\infty} c_k x_k$. The convergence of the series is understood to be with respect to the strong (norm) topology of X; in other words, $\|x - \sum_{k=-m}^{n} x_k\| \to 0$ as $m, n \to \infty$. A Shauder basis for a *Hilbert space* is a *Riesz basis* if it is equivalent to an orthonormal basis, that is, if it is obtained from an orthonormal basis by means of a bounded invertible operator (see Young [24]).

Let us now simply introduce the frame theory [24]. A function sequence $\{S_n\}_n \subset H$ is called a *frame* of a subspace H of $L^2(R)$ if there is a constant $C \geq 1$ such that

$$C^{-1}\|f\|^2 \leq \sum_n |\langle f, S_n\rangle|^2 \leq C\|f\|^2 \tag{3}$$

holds for any $f \in H$. A frame that ceases to be a frame when any one of its element is removed is said to be an *exact frame*. An exact frame is a *Riesz basis*. Obviously, a Riesz basis is also a frame. For any frame $\{S_n\}_n$ of H, there exists a so-called *dual frame* $\{\tilde{S}_n\}_n \subset H$ such that

$$f = \sum_n \langle f, \tilde{S}_n\rangle S_n = \sum_n \langle f, S_n\rangle \tilde{S}_n \tag{4}$$

holds in $L^2(R)$ for any $f \in H$. Take a linear operator T on H defined by

$$T(f) = \sum_n \langle f, S_n\rangle S_n.$$

Then the operator T is bounded, self-conjugate, and invertible due to the fact

$$\langle T(f), f\rangle = \sum_k |\langle f, S_n\rangle|^2$$

and the inequality (3). It is easy to see that the function sequence $\{T^{-1}(S_n)\}_n$ is a dual frame of the frame $\{S_n\}_n$. This T is called a *frame transform* of the frame $\{S_n\}_n$. The scalar sequence $\{\langle f, S_n\rangle\}_n$ is called a *moment sequence* of the function f to the frame $\{S_n\}_n$. Let $f = \sum_n c_n S_n$. If the scalar sequence $\{c_n\}_n$ is a moment sequence of a function to the frame $\{S_n\}_n$, then it must be $c_n = \langle T^{-1}(f), S_n\rangle$ for any $n \in Z$. This follows from the fact that $c_n = \langle h, S_n\rangle$ for some function $h \in H$, and in $L^2(R)$

$$T^{-1}(f) = \sum_n \langle h, S_n\rangle T^{-1}(S_n) = h.$$

Following are some notations used in this correspondence. For two measurable functions f and g on the real line R, let

$$\langle f, g\rangle = \int_R f(t)g(t)\,dt$$

$$\|f\| = \sqrt{\langle f, f\rangle}$$

$$\|f\|_* = \left(\int_0^{2\pi} |f(t)|^2\,dt\right)^{1/2}$$

$$\|f\|_\infty = \operatorname{ess\,sup} |f|$$

$$\|f\|_0 = \operatorname{ess\,inf} |f|.$$

II. General Sampling in Shift-Invariant Spaces

For a $\varphi \in L^2(R)$, the *shift-invariant space* [3], [12], [14] considered in this correspondence takes

$$V(\varphi) = \left\{ \sum_k c_k \varphi(\cdot - k) | \{c_k\}_k \in l^2 \right\} \subset L^2(R). \tag{5}$$

The φ is called a *generator* of $V(\varphi)$. In general, the function sequence $\{\varphi(\cdot - k)\}_k$ is not a Riesz basis of $V(\varphi)$. In fact, the function sequence $\{\varphi(\cdot - k)\}_k$ is a Riesz basis of $V(\varphi)$ if and only if

$$0 < \|G_\varphi\|_0 \leq \|G_\varphi\|_\infty < \infty$$

where $G_\varphi = \sum_k |\hat{\varphi}(\cdot + 2k\pi)|^2$, and $\hat{\varphi}$ is the *Fourier transform* of φ defined by

$$\hat{\varphi} = \int_R \varphi(t) \exp(-it\cdot)\,dt.$$

In this case, the generator is called *stable*. Moreover, the function sequence $\{\varphi(\cdot - k)\}_k$ is an orthonormal basis of $V(\varphi)$ if and only if $G_\varphi(\omega) = 1$ (a.e.).

It is easy to see that the stable shift-invariant space $V(\mathrm{sinc})$ is exactly the collection of all π-band signals of finite energy. Sampling in $V(\mathrm{sinc})$ is exactly the classical Shannon sampling theory for the band-limited signals of finite energy. Hereafter, we will study the sampling procedure in $V(\varphi)$ for a general continuous stable generator $\varphi \in L^2(R)$. We also need $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$. For any $f \in V(\varphi)$, there is a $\{c_k\}_k \in l^2$ such that $f = \sum_k c_k\varphi(\cdot - k)$ in $L^2(R)$. Since

$$\left|\sum_k c_k\varphi(\cdot - k)\right|^2 \le \sum_k |c_k|^2 \sum_k |\varphi(\cdot - k)|^2$$

the series $\sum_k c_k\varphi(\cdot - k)$ point-wise converges to a continuous function in $V(\varphi)$. Without loss of generality, we can take any $f \in V(\varphi)$ as a continuous function.

When we try to find an algorithm to reconstruct a continuous signal $f \in V(\varphi)$ by using its discrete samples $\{f(t_k)\}_k$, obviously the samples cannot be arbitrary; that is, some constraints should be imposed on the sampling points $\{t_k\}_k$ or the samples $\{f(t_k)\}_k$. The weaker the constraints are the better the reconstruction method is evaluated. Our objective in this section is to find a necessary and sufficient condition on the samples $\{f(t_k)\}_k$ such that a reconstruction formula like (1) holds. The result is formulated in the following theorem.

Theorem 2: Suppose that a continuous stable generator φ is such that $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$. Then there is a frame $\{S_n\}_n$ of $V(\varphi)$ such that $\{f(t_n)\}_n$ is a moment sequence to the frame $\{S_n\}_n$ and

$$f = \sum_n f(t_n)S_n \tag{6}$$

holds in $L^2(R)$ for any $f \in V(\varphi)$ if and only if there is a constant $C \ge 1$ such that

$$C^{-1}\|f\|^2 \le \sum_n |f(t_n)|^2 \le C\|f\|^2 \tag{7}$$

holds for any $f \in V(\varphi)$.

Proof:

Sufficiency.

Since $G_\varphi^{-1} \in L^2[0, 2\pi]$ is a 2π-periodic function, we can assume

$$G_\varphi^{-1}(\omega) = \sum_k g_k e^{-ik\omega}$$

in $L^2(R)$ with scalar sequence $\{g_k\} \in l^2$. Define ψ in $L^2(R)$ by

$$\hat{\psi} = \hat{\varphi}G_\varphi^{-1}.$$

Then we have

$$\psi = \sum_k g_k\varphi(\cdot - k) \in V(\varphi)$$

in $L^2(R)$. The fact $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$ shows that ψ can be chosen to be continuous. Moreover, the formula $\psi = \sum_k g_k\varphi(\cdot - k)$ also pointwise holds. Given $t_n \in R$, we have

$$\begin{aligned}
&\left(\sum_k |\psi(t_n - k)|^2\right)^{1/2}\\
&\quad= \frac{1}{\sqrt{2\pi}}\left\|\sum_k \psi(t_n - k)e^{ik\omega}\right\|_*\\
&\quad= \frac{1}{\sqrt{2\pi}}\left\|\sum_k\sum_l g_l\varphi(t_n - k - l)e^{ik\omega}\right\|_*\\
&\quad= \frac{1}{\sqrt{2\pi}}\left\|\sum_l g_l e^{-il\omega}\sum_k \varphi(t_n - k - l)e^{i(k+l)\omega}\right\|_*\\
&\quad= \frac{1}{\sqrt{2\pi}}\left\|G_\varphi^{-1}(\omega)\sum_k \varphi(t_n - k)e^{ik\omega}\right\|_*\\
&\quad\le \frac{1}{\sqrt{2\pi}}\left\|G_\varphi^{-1}(\omega)\right\|_\infty\left\|\sum_k \varphi(t_n - k)e^{ik\omega}\right\|_*\\
&\quad= \left\|G_\varphi^{-1}(\omega)\right\|_\infty\left(\sum_k |\varphi(t_n - k)|^2\right)^{1/2}.
\end{aligned}$$

Therefore, $\{\psi(t_n - k)\}_k \in l^2$ due to $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$. Let

$$q_\varphi(t_n, \cdot) = \sum_k \psi(t_n - k)\varphi(\cdot - k). \tag{8}$$

Then $q_\varphi(t_n, \cdot)$ is continuous in $V(\varphi)$. For any function $f \in V(\varphi)$, there is a scalar sequence $\{c_k\}_k \in l^2$ such that $f = \sum_k c_k\varphi(\cdot - k)$ holds in $L^2(R)$. Following the Parseval identity, we derive

$$\begin{aligned}
&\langle f, q_\varphi(t_n, \cdot)\rangle\\
&\quad= \frac{1}{2\pi}\left\langle \hat{f}, \hat{q}_\varphi(t_n, \cdot)\right\rangle\\
&\quad= \frac{1}{2\pi}\left\langle \hat{\varphi}\sum_k c_k e^{-ik\cdot}, \sum_k \hat{\varphi}\psi(t_n - k)e^{-ik\cdot}\right\rangle\\
&\quad= \frac{1}{2\pi}\left\langle |\hat{\varphi}|^2\sum_k c_k e^{-ik\cdot}, \sum_k\sum_l g_l\varphi(t_n - k - l)e^{-ik\cdot}\right\rangle\\
&\quad= \frac{1}{2\pi}\left\langle |\hat{\varphi}|^2\sum_k c_k e^{-ik\cdot}, \sum_l g_l e^{-ik\cdot}\sum_k \varphi(t_n - k)e^{-ik\cdot}\right\rangle\\
&\quad= \frac{1}{2\pi}\int_0^{2\pi} G_\varphi^{-1}(\omega)\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2\\
&\qquad\times\sum_k c_k e^{-ik\omega}\sum_k \varphi(t_n - k)e^{ik\omega}\,d\omega\\
&\quad= \frac{1}{2\pi}\int_0^{2\pi}\sum_k c_k e^{-ik\omega}\sum_k \varphi(t_n - k)e^{ik\omega}\,d\omega\\
&\quad= \sum_k c_k\varphi(t_n - k)\\
&\quad= f(t_n).
\end{aligned}$$

From (7), we derive

$$C^{-1}\|f\|^2 \le \sum_n |\langle f, q_\varphi(t_n, \cdot)\rangle|^2 \le C\|f\|^2.$$

It implies that $\{q_\varphi(t_n, \cdot)\}_n$ is a frame of $V(\varphi)$. For any $n \in Z$, take $S_n = T_q^{-1}(q_\varphi(t_n, \cdot))$, where T_q is the frame transform of the frame $\{q_\varphi(t_n, \cdot)\}_n$. Then $\{S_n\}_n$ is a dual frame of the frame $\{q_\varphi(t_n, \cdot)\}_n$ in $V(\varphi)$ and such that

$$f = \sum_n S_n(t)\langle f, q_\varphi(t_n, \cdot)\rangle = \sum_n f(t_n)S_n$$

holds in $L^2(R)$ for any $f \in V(\varphi)$. Meanwhile, for any $f \in V(\varphi)$ and any $n \in Z$, the fact

$$f(t_n) = \langle f, T_q(S_n)\rangle = \langle T_q(f), S_n\rangle$$

shows that $\{f(t_n)\}$ is a moment sequence to the frame $\{S_n\}_n$.

Necessity.

On the contrary, if there is a frame $\{S_n\}_n$ of $V(\varphi)$ such that $\{f(t_n)\}_n$ is a moment sequence to the frame $\{S_n\}_n$ and (6) holds in

$L^2(R)$ for any $f \in V(\varphi)$, then $f(t_n) = \langle T_s^{-1}(f), S_n\rangle$, where T_s is the frame transform of the frame $\{S_n\}_n$ (see Section I). Therefore,

$$\begin{aligned}\sum_n |f(t_n)|^2 &= \sum_n |\langle T_s^{-1}(f), S_n\rangle|^2 \\ &\le C\|T_s^{-1}(f)\|^2 \\ &\le C^2\|f\|^2\end{aligned}$$

and

$$\begin{aligned}\sum_n |f(t_n)|^2 &= \sum_n |\langle T_s^{-1}(f), S_n\rangle|^2 \\ &\ge C^{-1}\|T_s^{-1}(f)\|^2 \\ &\ge C^{-2}\|f\|^2\end{aligned}$$

hold for some constant $C \ge 1$. □

The frame $\{S_n\}_n$ of the shift-invariant space $V(\varphi)$ such that (6) holds in $L^2(R)$ for any $f \in V(\varphi)$ is called a *reconstruction frame*. In real-world application, we need to know the expression of the reconstruction frame $\{S_n\}_n$. Since T_q is the frame transform of the frame $\{q_\varphi(t_n, \cdot)\}_n$, we have

$$\begin{aligned}&T_q(q_\varphi(t_n, \cdot)) \\ &\quad= \sum_m \langle q_\varphi(t_n, \cdot), q_\varphi(t_m, \cdot)\rangle\, q_\varphi(t_m, \cdot) \\ &\quad= \frac{1}{2\pi}\sum_m \left\langle \sum_k \psi(t_n-k)\hat{\varphi}e^{-ik\cdot}, \right. \\ &\qquad\qquad \left. \sum_k \psi(t_m-k)\hat{\varphi}e^{-ik\cdot}\right\rangle q_\varphi(t_m, \cdot) \\ &\quad= \frac{1}{2\pi}\sum_m \left\langle |\hat{\varphi}|^2 \sum_k\sum_l g_l\varphi(t_n-k-l)e^{-ik\cdot}, \right. \\ &\qquad\qquad \left. \sum_k\sum_l g_l\varphi(t_m-k-l)e^{-ik\cdot}\right\rangle q_\varphi(t_m, \cdot) \\ &\quad= \frac{1}{2\pi}\sum_m \left\langle |\hat{\varphi}|^2 \left|\sum_l g_l e^{ik\cdot}\right|^2 \sum_k \varphi(t_n-k)e^{-ik\cdot}, \right. \\ &\qquad\qquad \left. \sum_k \varphi(t_m-k)e^{-ik\cdot}\right\rangle q_\varphi(t_m, \cdot) \\ &\quad= \frac{1}{2\pi}\sum_m q_\varphi(t_m, \cdot)\int_0^{2\pi} G_\varphi^{-1}(\omega)\sum_k \varphi(t_n-k)e^{-ik\omega} \\ &\qquad \times \sum_k \varphi(t_m-k)e^{ik\omega}\, d\omega.\end{aligned}$$

Let

$$A_{nm} = \int_0^{2\pi} G_\varphi^{-1}(\omega)\sum_k \varphi(t_n-k)e^{-ik\omega}\sum_k \varphi(t_m-k)e^{ik\omega} d\omega.$$

Then we have

$$q_\varphi(t_n, \cdot) = \frac{1}{2\pi}\sum_m A_{nm}S_m.$$

Since T_q is a frame transform, the infinite-dimensional matrix $A = (A_{nm})$ is invertible. Denote by $A^{-1} = (d_{mn})$ the inverse matrix of A. Then we derive

$$S_m = 2\pi \sum_n d_{mn}q_\varphi(t_n, \cdot)$$

for any $m \in Z$. This is formulated in the following corollary.

Corollary 1: In Theorem 2, the reconstruction frame $\{S_n\}_n$ is derived by

$$S_n = 2\pi \sum_n d_{mn}q_\varphi(t_m, \cdot) \tag{9}$$

where (d_{mn}) is the inverse matrix of the infinite-dimensional matrix (A_{mn}), and

$$A_{mn} = \int_0^{2\pi} G_\varphi^{-1}(\omega)\sum_k \varphi(t_n-k)e^{-ik\omega}\sum_k \varphi(t_m-k). \tag{10}$$

In the following sections, we will explore the relation between the matrix (A_{mn}) and the reconstruction frame $\{S_n\}_n$ in more detail.

III. Regular Sampling in Shift-Invariant Spaces

An important case of sampling is the so-called regular sampling, i.e., $t_n = n$ for all $n \in Z$. As we mentioned in the Introduction, many people have contributed to this area of research. The following theorem contributes in a different direction. Since it is not so difficult to understand, we just give a swift proof here. We will also need the notation $\hat{\varphi}^*$ defined by

$$\hat{\varphi}^* = \sum_k \varphi(k)\exp(ik\cdot).$$

Obviously, $\hat{\varphi}^*$ is a 2π-periodic function in $L^2[0, 2\pi]$.

Theorem 3: Let φ be a continuous stable generator such that $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$. Then the following three statements are equivalent.

1) $0 < \|\hat{\varphi}^*\|_0 \le \|\hat{\varphi}^*\|_\infty < \infty$.
2) There is a frame $\{S_n\}_n$ of $V(\varphi)$ such that $\{f(n)\}_n$ is a moment sequence to the frame $\{S_n\}_n$ and $f = \sum_n f(n)S_n$ for any $f \in V(\varphi)$.
3) There is a Riesz basis $\{S_n\}_n$ of $V(\varphi)$ such that $f = \sum_n f(n)S_n$ for any $f \in V(\varphi)$.

In these cases, the $\{S_n\}_n$ is defined by the formulas $S_n = S_0(\cdot - n)$ (a.e.) and $\hat{S}_0 = \hat{\varphi}/\hat{\varphi}^*$ (a.e.).

Proof: For any $f \in V(\varphi)$, there is a scalar sequence $\{c_k\}_k \in l^2$ such that $f = \sum_k c_k\varphi(\cdot - k)$ holds both in $L^2(R)$ and pointwise. Since

$$\begin{aligned}\|f\|^2 &= \frac{1}{2\pi}\left\|\hat{f}\right\|^2 \\ &= \frac{1}{2\pi}\left\|\hat{\varphi}\sum_k c_k e^{-ik\cdot}\right\|^2 \\ &= \frac{1}{2\pi}\left\|G_\varphi^{1/2}\sum_k c_k e^{-ik\cdot}\right\|_*^2,\end{aligned}$$

and

$$\begin{aligned}\sum_k |f(k)|^2 &= \frac{1}{2\pi}\left\|\sum_k\sum_l c_l\varphi(k-l)e^{-ik\cdot}\right\|_*^2 \\ &= \frac{1}{2\pi}\left\|\hat{\varphi}^*\sum_k c_k e^{-ik\cdot}\right\|_*^2\end{aligned}$$

we can conclude that the fact

$$C^{-1}\|f\|^2 \le \sum_k |f(k)|^2 \le C\|f\|^2$$

for any $f \in V(\varphi)$ is equivalent to the fact

$$C^{-1}\|G_\varphi^{1/2}\|_0 \le |\hat{\varphi}^*| \le C\|G_\varphi^{1/2}\|_\infty.$$

By Theorem 2, these arguments show the equivalence of *statement 1)* and *statement 2)*.

Consider the q_φ defined by(8). For any scalar sequence $\{c_k\} \in l^2$, we have

$$\begin{aligned}\left\|\sum_k c_k q_\varphi(k,\cdot)\right\|^2 &= \frac{1}{2\pi}\left\|\hat{\varphi}\sum_k \psi(n-k)c_k e^{-ik\cdot}\right\|^2\\ &= \frac{1}{2\pi}\left\|G_\varphi^{1/2}\sum_l g_l e^{ik\cdot}\sum_k \varphi(n-k)c_k e^{-ik\cdot}\right\|_*^2\\ &= \frac{1}{2\pi}\left\|G_\varphi^{-1/2}\hat{\varphi}^*\sum_k c_k e^{-ik\cdot}\right\|_*^2.\end{aligned}$$

Then *statement 1)* shows

$$\begin{aligned}\|\hat{\varphi}^*\|_0^2\|G_\varphi\|_0\sum_k |c_k|^2 &\le \left\|\sum_k c_k q_\varphi(k,\cdot)\right\|^2\\ &\le \|\hat{\varphi}^*\|_\infty^2\|G_\varphi\|_\infty\sum_k |c_k|^2\end{aligned}$$

for any scalar sequence $\{c_k\}_k \in l^2$. It implies that the function sequence $\{q_\varphi(n,\cdot)\}_n$ is a Riesz basis of $V(\varphi)$. Then there is a unique dual Riesz basis $\{S_n\}_n$ of the Riesz basis $\{q_\varphi(n,\cdot)\}_n$ biorthogonal to $\{q_\varphi(n,\cdot)\}_n$ such that *statement 3)* holds. Since *statement 3)* implies *statement 2)*, thereafter, *statement 1)* automatically, it shows the equivalence of *statement 1)* and *statement 3)*.

Finally, the formula $S_n = S_0(\cdot - n)$ and $\hat{S}_0 = \hat{\varphi}/\hat{\varphi}^*$ can follow from [9] and [19]. But we would like to use Corollary 1 to derive it so as to show the potential of Corollary 1. From (10), we have

$$A_{nm} = \int_0^{2\pi} \frac{|\hat{\varphi}^*|^2}{G_\varphi} e^{-i(n-m)\omega}\,d\omega.$$

Suppose that $\{c_k\}_k$ are the Fourier coefficients of the 2π-periodic function $\frac{|\hat{\varphi}^*|^2}{G_\varphi}$ in $L^2[0, 2\pi]$. Then, we derive $A_{nm} = c_{n-m}$ for any $m, n \in Z$, and

$$\sum_m c_{n-m}\hat{S}_m = \hat{q}_\varphi(n,\cdot) = \frac{|\hat{\varphi}^*|^2}{G_\varphi} \times \frac{\hat{\varphi}}{\hat{\varphi}^*}e^{-in\cdot}.$$

Suppose that the scalar sequence $\{d_n\}_n$ is the Fourier coefficients of the 2π-periodic function $\frac{G_\varphi}{|\hat{\varphi}^*|^2}$ in $L^2[0, 2\pi]$. Make an infinite-dimensional matrix $D = (d_{m-n})$. Then the matrix D is the inverse matrix of the matrix A, because the formula $\sum_n c_{k-n}d_{n-j} = \delta_{k-j}$ is equivalent to the formula

$$\sum_k c_k e^{ik\omega}\sum_n d_n e^{in\omega}e^{ij\omega} = e^{ij\omega} \tag{11}$$

and (11) automatically holds. The δ_{k-j} used here is the Dirac function, which takes value 1 whenever $k = j$ and takes value 0 otherwise. Define the function S_0 by the formula $\hat{S}_0 = \hat{\varphi}/\hat{\varphi}^*$ in $L^2(R)$. From (11), we have

$$\hat{S}_m = \frac{|\hat{\varphi}^*|^2}{G_\varphi}\hat{S}_0\sum_k d_{m-n}e^{-in\cdot} = \hat{S}_0 e^{-im\cdot}.$$

It shows $S_m = S_0(\cdot - m)$ for any $m \in Z$. □

This theorem implies that for regular sampling in shift-invariant spaces, the reconstruction frame $\{S_n\}_n$ is an exact frame, i.e., a Riesz basis. This fact is also determined by the samples $\{\varphi(k)\}_k$ of the generator φ, i.e., $C^{-1} \le |\hat{\varphi}^*| \le C$ for some constant $C \ge 1$.

When φ is a stable continuous *cardinal generator*, that is, $\hat{\varphi}^* = 1$ (a.e.). Then $\hat{S}_0 = \hat{\varphi}$, i.e., $S_0 = \varphi$. Therefore, the reconstruction formula becomes $f = \sum_k f(k)\varphi(\cdot - k)$ for any $f \in V(\varphi)$. By the way, the fact that *statement 1)* implies *statement 3)* is also found by Walter [19] when $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1$. The formula $S_m = S_0(\cdot - m)$ for any $m \in Z$ has also been found by Walter [19], Aldroubi and Unser [3], and Chen and Itoh [9] under their assumed conditions on the generator φ.

Take $\varphi = \mathrm{sinc}$. Then φ is a stable continuous cardinal generator. The shift-invariant space $V(\varphi)$ is the collection of all π-band signals of finite energy. Then, the classical Shannon sampling formula

$$f = \sum_k f(k)\frac{\sin\pi(\cdot - k)}{\pi(\cdot - k)}$$

automatically follows from Theorem 3.

IV. Perturbation of Regular Sampling in Shift-Invariant Spaces

Another important case of sampling is the perturbation of regular sampling, $t_n = n + r_n$ $(r_n \in (-1, 1))$ for any $n \in Z$. A fundamental question in this case is to estimate the range of the perturbation $\{r_k\}_k$. Following Kadec's theorem for the band-limited signals of finite energy, we have found an estimate for perturbation in the setting of wavelet subspaces by using Riesz basis theory. In the following, we will derive a sharp estimate by using frame theory.

In order to establish the algorithm for perturbation of regular sampling in shift-invariant spaces, we need to introduce the function class $L_\sigma^\lambda[a, b]$ $(\lambda > 0, \sigma \in [0, 1), 0 \in [a, b] \subset [-1, 1])$ defined and used in our previous work [6]. We have reasoned that this class is an appropriate collection by giving some propositions in that paper. Here we will recall the definition and give an additional proposition, which are useful to the piecewise differentiable stable generators in many real applications.

Definition 1: $L_\sigma^\lambda[a, b]$ $(\lambda > 0, \sigma \in [0, 1)$ and $0 \in [a, b] \subset [-1, 1])$ consists of all the measurable functions f, for which the norm

$$\|f\|_{L_\sigma^\lambda[a,b]} = \sup_{\{r_k\}_k \subset [a,b]} \frac{\sum_k |f(k+\sigma+r_k) - f(k+\sigma)|}{\sup_k |r_k|^\lambda} < \infty.$$

The following proposition is very useful to the piecewise differentiable generators in many real applications.

Proposition 1: Let function f on R be differentialable on the intervals $k + \sigma + [a, b]$ $(\sigma \in [0, 1), 0 \in [a, b] \subset [-1, 1])$ for any $k \in Z$. If the series $\sum_k \sup_{[a,b]} |f'(\cdot + k + \sigma)| < \infty$ then $f \in L_\sigma^1[a, b]$, where f' is the derivative of f.

Proof: Since

$$\begin{aligned}\sum_k |f(k+\sigma+r_k) - f(k+\sigma)| &= \sum_k \left|\int_0^{r_k} f'(k+\sigma+t)\,dt\right|\\ &\le \sum_k \int_0^{\sup_k |r_k|} |f'(k+\sigma+t)|dt\\ &\le \sup_k |r_k| \sum_k \sup_{[a,b]} |f'(k+\sigma+\cdot)|\end{aligned}$$

it implies $f \in L_\sigma^1[a, b]$. □

Now we show the theorem for perturbation of regular sampling in shift-invariant spaces. We will also discuss some special cases after the proof.

Theorem 4: Suppose that φ is a continuous stable generator such that $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$, and

1) $0 < \|\hat{\varphi}^*\|_0 \le \|\hat{\varphi}^*\|_\infty < \infty$.
2) $\varphi \in L_0^\lambda[a, b]$.

Then for any $\{r_k\}_k \subset [-r_\varphi, r_\varphi] \cap [a, b]$, there is a frame $\{S_k\}_k$ of $V(\varphi)$ such that the scalar sequence $\{f(k+r_k)\}_k$ is a moment sequence to the frame $\{S_k\}_k$ and $f = \sum_k f(k+r_k)S_k$ holds in $L^2(R)$ for any $f \in V(\varphi)$ if

$$r_\varphi < \left(\frac{\|\hat{\varphi}^*(\omega)\|_0}{\|\varphi\|_{L_0^\lambda[a,b]}}\right)^{1/\lambda}. \tag{12}$$

Proof: Let $t_k = k + r_k$. By Theorem 2, we only need to show that there is a constant $C \ge 1$ such that

$$C^{-1}\|f\|^2 \le \sum_k |f(t_k)|^2 \le C\|f\|^2$$

holds for any $f \in V(\varphi)$. If we can show that there is a positive number $\theta < 1$ such that

$$\sum_k |f(t_k) - f(k)|^2 \le \theta^2 \sum_k |f(k)|^2 \tag{13}$$

holds for any $f \in V(\varphi)$, then

$$(1-\theta)^2 \sum_k |f(k)|^2 \le \sum_k |f(t_k)|^2 \le (1+\theta)^2 \sum_k |f(k)|^2.$$

By Theorem 3, *statement 1)* implies that there is a constant $C \ge 1$ such that

$$C^{-1}\|f\|^2 \le \sum_k |f(k)|^2 \le C\|f\|^2$$

for any $f \in V(\varphi)$. These arguments show that

$$C^{-1}(1-\theta)^2\|f\|^2 \le \sum_k |f(t_k)|^2 \le C(1+\theta)^2\|f\|^2$$

for any $f \in V(\varphi)$. By Theorem 2, Theorem 4 is shown.

All that remains to be shown is (13). For an $f \in V(\varphi)$, there is a scalar sequence $\{c_k\}_k \in l^2$ such that $f = \sum_k c_k\varphi(\cdot - k)$ holds both in $L^2(R)$ and pointwise. Let

$$\begin{aligned}\Delta &= \sum_k |f(t_k) - f(k)|^2 \\ &= \sum_k \left|\sum_l c_l(\varphi(t_k - l) - \varphi(k-l))\right|^2 \\ &= \sum_n \sum_{k,l} c_k c_l(\varphi(t_n - k) - \varphi(n-k)) \\ &\quad \times (\varphi(t_n - l) - \varphi(n-l)) \\ &= \sum_{k,l} c_k c_l \sum_n (\varphi(t_n - k) - \varphi(n-k)) \\ &\quad \times (\varphi(t_n - l) - \varphi(n-l)).\end{aligned}$$

Take

$$a_{k,l} = \sum_n (\varphi(t_n - k) - \varphi(n-k))(\varphi(t_n - l) - \varphi(n-l)).$$

Then $a_{k,l} = a_{l,k}$ holds for any $k, l \in Z$. Following the argument in [6], we have

$$\Delta = \sum_{k,l} a_{kl}c_k c_l \le \left(\sum_k c_k^2\right) \sup_k \sum_l |a_{kl}|$$

and

$$\sup_k \sum_l |a_{kl}| \le \left(\|\varphi\|_{L_0^\lambda[a,b]} \sup_\beta |r_\beta|^\lambda\right)^2.$$

Hence

$$\Delta \le \left(\sum_k c_k^2\right)(\|\varphi\|_{L_0^\lambda[a,b]} \sup_\beta |r_\beta|^\lambda)^2.$$

Recall the proof of Theorem 3. We know

$$\|\hat{\varphi}^*\|_0^2 \sum_k c_k^2 \le \sum_k |f(k)|^2.$$

Therefore, we only need to show

$$\left(\sum_k c_k^2\right)\left(\|\varphi\|_{L_0^\lambda[a,b]} \sup_\beta |r_\beta|^\lambda\right)^2 \le \theta^2\|\hat{\varphi}^*\|_0^2 \sum_k c_k^2$$

that is, $\|\varphi\|_{L_0^\lambda[a,b]} \sup_\beta |r_\beta|^\lambda \le \theta\|\hat{\varphi}^*\|_0$. This is exactly implied by (12). □

Once again, we are interested in the expression of the reconstruction frame $\{S_n\}_n$. By Corollary 1

$$\begin{aligned}A_{nm} &= \int_0^{2\pi} G_\varphi^{-1} \sum_k \varphi(n + r_n - k)e^{-ik\omega} \\ &\quad \times \sum_k \varphi(m + r_m - k)e^{ik\omega}\,d\omega \\ &= \int_0^{2\pi} G_\varphi^{-1} \sum_l \varphi(l + r_n)e^{il\omega} \\ &\quad \times \sum_l \varphi(l + r_m)e^{-il\omega}e^{i(m-n)\omega}\,d\omega.\end{aligned}$$

Let $m - n = k$. Then

$$A_{n(n+k)} = \int_0^{2\pi} G_\varphi^{-1} \sum_l \varphi(l+r_n)e^{il\omega} \sum_l \varphi(l+r_{n+k})e^{-il\omega}e^{ik\omega}\,d\omega.$$

Let

$$B_{nk}(\omega) = G_\varphi^{-1}(\omega)\sum_l \varphi(l+r_n)e^{il\omega} \sum_l \varphi(l + r_{n+k})e^{-il\omega}.$$

Then we have the formulas

$$A_{n(n+k)} = \int_0^{2\pi} B_{nk}(\omega)e^{ik\omega}(\omega)\,d\omega$$

and

$$\begin{aligned}\sum_k A_{nm}e^{imt} &= \sum_k A_{n(n+k)}e^{i(n+k)t} \\ &= e^{int}\sum_k e^{ikt}\int_0^{2\pi} B_{nk}(\omega)e^{ik\omega}\,d\omega.\end{aligned}$$

Let

$$C_n = \sum_k e^{ikt}\int_0^{2\pi} B_{nk}(\omega)e^{ik\omega}\,d\omega.$$

Then $\{C_n\}_n$ are the eigenvalues of the matrix A. Since the matrix A is invertible with the inverse matrix $A^{-1} = (C_n^{-1}r_{n-m})$, from Corollary 1, we have $\hat{S}_n = 2\pi C_n^{-1}\hat{q}_\varphi(n + r_n, \cdot)$ for any $n \in Z$. Simple calculation shows that the formula

$$\hat{q}_\varphi(n + r_n, \cdot) = e^{-in\cdot}G_\varphi^{-1}\hat{\varphi}\sum_l \varphi(l + r_n)e^{il\cdot}$$

holds for any $n \in Z$. Finally, we get the formula to calculate the reconstruction frame $\{S_n\}_n$ as

$$\hat{S}_n = \frac{2\pi\sum_k \varphi(k+r_n)e^{ik\cdot}}{C_n G_\varphi}\hat{\varphi}e^{-in\cdot}.$$

In a regular sampling case, $r_n = 0$ for any $n \in Z$, and $C_n = G_\varphi/|\hat{\varphi}^*|^2$. Then we have $\hat{S}_n = e^{-in\omega}\hat{\varphi}/\hat{\varphi}^*$ and $\hat{S}_0 = \hat{\varphi}/\hat{\varphi}^*$, that is, exactly the result of Theorem 3.

The estimate (12) is also derived in our previous work [6] when the continuous stable generator φ is orthonormal and satisfies $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1$. But in Theorem 4, the estimate (12) also holds for a nonorthonormal continuous stable generator whenever it satisfies $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$. The authors [6] also derived an estimate for perturbation of regular sampling in shift-invariant spaces with a nonorthonormal continuous stable generator when φ satisfies $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1$. But that estimate of [6] is somewhat awkward and rough. The examples in Section VI will make it more clear.

It is also worth noting that the reconstruction frame may turn out to be a Riesz basis under the assumption of Theorem 4. It is also possible to derive the result of Theorem 4 directly by Riesz basis theory. But we do not touch Riesz basis in the preceding argument, and derive the result swiftly from Theorem 2. We leave the Riesz basis discussion to future work.

V. Shift Sampling in Shift-Invariant Spaces

Unfortunately, there are some important continuous stable generators φ's with $\|\hat{\varphi}^*\|_0 = 0$. An obvious example is the B-spline of degree 2, which has been calculated in our previous works [4]–[6]. As done by Janssen [13] for Walter's sampling theorem [19], and the authors [6] for irregular sampling theorem, we also treat it by shift sampling. Then the shift-sampling versions of sampling theorems can be attained by using the Zak transform $Z_\varphi(\sigma, \cdot)$ $(\sigma \in [0, 1))$ defined by

$$Z_\varphi(\sigma, \cdot) = \sum_n \varphi(\sigma + n) \exp(in\cdot). \tag{14}$$

Since the proofs of the shift sampling theorems are very similar to those of Theorems 3 and 4, respectively, we here only to give the statements of the shift sampling theorems by omitting the proofs.

Theorem 5: Suppose φ is a continuous stable generator such that $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$. Then, for a $\sigma \in [0, 1)$, the following three items are equivalent.

1) $0 < \|Z_\varphi(\sigma, \cdot)\|_0 \le \|Z_\varphi(\sigma, \cdot)\|_\infty < \infty$.
2) There is a frame $\{S_{\sigma, n}\}_n$ of $V(\varphi)$ such that $\{f(n+\sigma)\}_n$ is a moment sequence to the frame $\{S_{\sigma, n}\}_n$ and
$$f = \sum_n f(n+\sigma) S_{\sigma, n}$$
holds in $L^2(R)$ for any $f \in V(\varphi)$.
3) There is a Riesz basis $\{S_{\sigma, n}\}_n$ of $V(\varphi)$ such that
$$f = \sum_n f(n) S_n$$
holds in $L^2(R)$ for any $f \in V(\varphi)$.

In these cases, the reconstruction frame $\{S_{\sigma, n}\}_n$ is derived by the formulas $S_{\sigma, n} = S_{\sigma, 0}(\cdot - n)$ and $\hat{S}_{\sigma, 0} = \hat{\varphi}/Z_\varphi(\sigma, \cdot)$.

The following is the shift-sampling version of the algorithm for perturbation of regular sampling in shift-invariant spaces.

Theorem 6: Suppose φ is a continuous stable generator such that $\varphi = O((1+|\cdot|)^{-s})$ for some $s > 1/2$, and

1) $0 < \|Z_\varphi(\sigma, \cdot)\|_0 \le \|Z_\varphi(\sigma, \cdot)\|_\infty < \infty$.
2) $\varphi \in L^\lambda_\sigma[a, b]$.

Then, for any $\{r_k\}_k \subset [-r_{\sigma,\varphi}, r_{\sigma,\varphi}] \cap [a, b]$, there is a frame $\{S_{\sigma, k}\}_k$ of $V(\varphi)$ such that $\{f(k+\sigma+r_k)\}_k$ is a moment sequence to the frame $\{S_{\sigma, k}\}_k$ and

$$f = \sum_k f(k+\sigma+r_k) S_{\sigma, k}$$

holds in $L^2(R)$ for any $f \in V(\varphi)$ if

$$r_{\sigma,\varphi} < \left(\frac{\|Z_\varphi(\sigma, \cdot)\|_0}{\|\varphi\|_{L^\lambda_\sigma[a,b]}} \right)^{1/\lambda}. \tag{15}$$

VI. Examples to Show the Algorithm

Since the Daubechies mother wavelets and the Meyer mother wavelet are all the orthonormal continuous stable generators [11], [17], [21], the estimates by theorems in this correspondence are the same to those derived by our previous works (see the discussion after Theorem 4). We here calculate the B-spline of degree 1 (denoted by N_1). We derive the estimate $r_{N_1} < 1/2$, which is better than our previous estimate $r_{N_1} < 1/(2\sqrt{3})$, and which is shown by Liu and Walter to be optimal [16]. Unfortunately, we are not yet sure that the estimate

$$\sup_{\sigma \in (-1,1)} \frac{\|Z_\varphi(\sigma, \omega)\|_0}{\|\varphi\|_{L^\lambda_\sigma[a,b]}}$$

is optimal for a general continuous stable generator; and a question remains of what is the optimal estimate for a general continuous stable generator.

Example 1 [10]: Take the B-spline of degree 1

$$N_1(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)}.$$

Then $\hat{N}_1^* = e^{i\omega}$. Since $\|N_1\|_{L^1_0[-1,1]} = 3$, we derive the estimate $r_{N_1} < 1/3$. Suppose $r_k \ge 0$ (or $r_k \le 0$) for all $k \in Z$. Then

$$\|N_1\|_{L^1_0[-1,0]} = \|N_1\|_{L^1_0[0,1]} = 2.$$

Therefore, the estimate is $r_{N_1} < 1/2$. Liu and Walter [16] have shown that the estimate $r_{N_1} < 1/2$ is optimal when $r_k \ge 0$ (or $r_k \le 0$) for all $k \in Z$. Compared to the estimate $r_{N_1} < 1/(2\sqrt{3})$ derived in our previous work, the present result does improve our previous one [6].

Again, we want to know the structure of the reconstruction frame. For regular sampling in $V(N_1)$, from Theorem 3 we have $\hat{S}_n = \hat{N}_1 e^{-in\omega}/e^{i\omega}$. Therefore, $S_n = N_1(\cdot - n - 1)$ for any $n \in Z$.

For perturbation of regular sampling in $V(N_1)$, from Corollary 1, we have

$$\hat{S}_n = 2\pi \hat{N}_1 e^{-in\omega} \sum_k N_1(k + r_n) e^{ik\omega} / (C_n G_{N_1}).$$

Suppose $r_n \ge 0$ for all $n \in Z$. Then

$$\begin{aligned} C_n &= \sum_k e^{ikt} \int_0^{2\pi} \\ &\quad \times \frac{\sum_l N_1(l+r_n)e^{il\omega} \sum_l N_1(l+r_{n+k})e^{il\omega}}{G_{N_1}} e^{ik\omega}\, d\omega \\ &= \sum_k e^{ikt} \int_0^{2\pi} \\ &\quad \times \frac{(r_n + (1+r_n)e^{i\omega})(r_{n+k} + (1+r_{n+k})e^{i\omega})}{G_{N_1}} e^{ik\omega}\, d\omega. \end{aligned}$$

Let $u_{nk} = \int_0^{2\pi} (r_n + (1+r_n)e^{i\omega}) G_{N_1}^{-1} e^{ik\omega} d\omega$. Then

$$\begin{aligned} C_n(\omega) &= \sum_k e^{ik\omega} u_{nk} r_{n+k} + \sum_k e^{ik\omega} u_{n(k+1)}(1 + r_{n+k}) \\ &= 2\pi \frac{r_n + (1+r_n)e^{i\omega}}{G_{N_1}} e^{-i\omega} \\ &\quad + \sum_k e^{ik\omega} u_{nk}(r_{n+k} + r_{n+k-1}e^{-i\omega}). \end{aligned}$$

Let

$$B_n = \sum_k (r_{n+k} + r_{n+k-1} e^{-i\omega}) u_{nk} e^{ik\omega}.$$

Since

$$G_{N_1} = |\sin(\cdot/2)|^4 \sum_k |\cdot/2 + k\pi|^{-4}$$

finally we have

$$\hat{S}_n = \frac{2\pi(r_n + (1+r_n)e^{i\omega})\hat{N}_1 e^{-in\omega}}{2\pi(r_n e^{-i\omega} + (1+r_n)) + G_{N_1} B_n}.$$

The next example shows the importance of shift sampling theorem.

Example 2 [10]: Take the B-spline of degree 2

$$N_2(t) = \frac{t^2}{2}\chi_{[0,1)} + \frac{6t - 2t^2 - 3}{2}\chi_{[1,2)} + \frac{(3-t)^2}{2}\chi_{[2,3)}.$$

Then $\hat{N}_2^*(\omega) = e^{i\omega}(e^{i\omega} + 1)/2 = 0$ at the point $\omega = \pi$. So we have to use the shift sampling theorem. The Zak transform of N_2 is

$$Z_{N_2}(1/2, \omega) = 1/8 + 3e^{i\omega}/4 + e^{2i\omega}/8.$$

The facts

$$|Z_{N_2}(1/2, \omega)| \geq 3/4 - 1/8 - 1/8 = 1/2$$

and

$$Z_{N_2}(1/2, \pi) = 1/2$$

imply $\|Z_{N_2}(\frac{1}{2}, \cdot)\|_0 = 1/2$. Since the derivative

$$N_2'(t) = t\chi_{[0,1)} + (3-4t)\chi_{[1,2)} + (3-t)\chi_{[2,3)}$$

by Proposition 1, we have $\|N_2\|_{L^1_{1/2}[-1/4,1/4]} = 2$. Finally, we derive the estimate $r_{1/2,N_2} < 1/4$.

For regular sampling in $V(N_2)$, by Theorem 3, the reconstruction frame $\{S_n\}_n$ is determined by $\hat{S}_n = 8\hat{N}_2 e^{-in\omega}/(1+6e^{i\omega}+e^{2i\omega})$ for any $n \in Z$. For perturbation of regular sampling in $V(N_2)$, since the calculation to find an explicit expression of the reconstruction frame $\{S_n\}_n$ is very complicated, we leave it to future work. Anyway, the computer-based solution can follow from Corollary 1. Shifting by different σ, the estimate derived by shift sampling theorem is, in general, different. To find a shift to derive an optimal estimate is a useful and interesting future work.

The Daubechies mother wavelets, the Meyer mother wavelet, and the B-splines are all scaling functions of some MRAs. But the next continuous stable generator ceases to be a scaling function of an MRA.

Example 3: Take the Gaussian kernel $K = 1/\sqrt{2\pi}\, e^{-t^2/2}$. Then $\hat{K} = e^{-\omega^2/2}$. Therefore, K is not a scaling function of any MRA since $\hat{K}(2k\pi) \neq 0$ for any $k \neq 0$. However,

$$G_K = \sum_k e^{-(\omega+2k\pi)^2} = e^{-\omega^2} + 2\sum_{k=1}^{\infty} e^{-(\omega+2k\pi)^2}$$

For any $\omega \in [0, 2\pi]$, we derive

$$G_K \geq e^{-4\pi^2} + 2\int_2^{\infty} e^{-x^2}\, dx = e^{-4\pi^2} + \sqrt{2\pi}/e^2$$

and

$$G_K \leq 1 + 2\int_0^{\infty} e^{-x^2}\, dx = 1 + \sqrt{2\pi}.$$

It shows that K is a stable generator of the shift-invariant space $V(K)$. By Poisson summation formula, we know that

$$\hat{K}^* = 1/\sqrt{2\pi}\sum_k e^{-(\omega+2k\pi)^2/2}.$$

Then we have $e^{-2\pi}/\sqrt{2\pi} + 2/e \leq \hat{K}^* \leq 1/\sqrt{2\pi} + 2$. From Theorem 3, for regular sampling in $V(K)$, the reconstruction frame $\{S_n\}_n$ is given by

$$\hat{S}_n = \sqrt{2\pi}e^{-\omega^2/2}e^{-in\omega} / \sum_k e^{-(\omega+2k\pi)^2/2}.$$

For perturbation of regular sampling in $V(K)$, the fact $K' = 1/\sqrt{2\pi} t e^{t^2/2}$ implies $K \in L_0^1[-1, 1]$ and

$$\|K\|_{L_0^1[-1,1]} \leq 1/\sqrt{2\pi}(\sqrt{2}e^{-1/4} + 4e^{-1/2}).$$

Therefore, we derive an estimate

$$r_K < (e^{1/2-2\pi} + 2e^{-1/2}\sqrt{2\pi})/(\sqrt{2}e^{1/4} + 4) \approx 0.35.$$

The right-hand side is larger than $1/4$. It is difficult to give an explicit expression of the reconstruction frame $\{S_n\}_n$. But a computer-based solution of $\{S_n\}_n$ can be derived from Corollary 1.

The next example is calculated to compare to the traditional Kadec's theorem.

Example 4: Take the Shannon sampling function $\mathrm{sinc} = \frac{\sin\pi(\cdot)}{\pi(\cdot)}$. Then $\hat{\mathrm{sinc}} = \chi_{[-\pi,\pi]}$, $\hat{\mathrm{sinc}}^* = 1$, and $G_{\mathrm{sinc}} = 1$. This implies that sinc is a continuous cardinal orthonormal stable generator of the shift-invariant space $V(\mathrm{sinc})$. For regular sampling in $V(\mathrm{sinc})$, the reconstruction frame $\{S_n\}_n$ is given by $S_n = \sin(\pi(\cdot - n))/(\pi(\cdot - n))$. For perturbation of regular sampling in $V(\mathrm{sinc})$, Theorem 4 cannot be used, since the series $\sum_k |\sin(r_k\pi)/((k + r_k)\pi)|$ is generally divergent. To find a theorem like Theorem 4 that covers the Shannon sampling function is also a very interesting open problem. At present , however, we can only use Theorem 2. For any $f \in V(\mathrm{sinc})$, a simple calculation shows

$$\Delta = \sum_k |f(k + r_k) - f(k)|^2$$

$$\leq (1 - \cos\pi r + \sin\pi r)^2 \sum_k |f(k)|^2$$

where $r = r_{\mathrm{sinc}}$. Since $1 - \cos\pi r + \sin\pi r < 1$ when $r < 1/4$, it deduces the estimate $r_{\mathrm{sinc}} < 1/4$. Kadec [24] has shown that the estimate $r_{\mathrm{sinc}} < 1/4$ is optimal. The reconstruction frame can be derived from Corollary 1.

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