

On Simple Oversampled A/D Conversion in Shift-Invariant Spaces

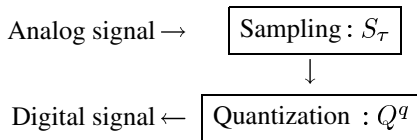
Wen Chen, *Member, IEEE*, Bin Han, and Rong-Qing Jia

Abstract—It has been found that the quantization error e for a conventional oversampled analog-to-digital (A/D) conversion behaves like $\|e\|^2 = O(\tau^2)$ with respect to the sampling rate τ . Recently, conventional A/D conversion has been extended to A/D conversion based on shift-invariant spaces. As consequences of such extension, it offers rich choices to build a nonideal A/D conversion system of high accuracy and low computational complexity, as well as reduces the noise sensitivity and computational complexity in digital-to-analog (D/A) conversion. Therefore, it is necessary to establish the estimate of quantization error for the extended A/D conversion based on shift-invariant spaces. In this paper, we introduce a constructive method to establish an estimate of the quantization error as $|e|^2 = O(\tau^2)$ for oversampled A/D conversion in shift-invariant spaces. Meanwhile, we demonstrate that the bit rate required to encode the converted digital signal in such A/D conversion scheme only increases as the logarithm of the sampling ratio. Therefore, the quantization error is an exponentially decaying function of the bit rate. In order to establish such an estimate, we need the nonuniform sampling theorem for shift-invariant spaces, which, as the necessary preparation, is studied prior to introducing the constructive method.

Index Terms—Analog-to-digital (A/D) conversion, bit rate, generator, prefiltering, quantization error, reconstruction frame, sampling, shift-invariant space.

I. INTRODUCTION

DIGITIZING an analog signal requires discretization in both time and amplitude. In a simple analog-to-digital (A/D) conversion, the signal is first discretized in time using regular sampling with a sampling interval τ , followed by a uniform scalar quantization with a quantization step q as illustrated by



A signal f is of *finite energy* if $\|f\| < \infty$, where $\|f\|$ is the *square norm* of f defined by

$$\|f\| = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2}. \quad (1)$$

We also denote by $L^2(\mathbb{R})$ the signal space of finite energy, that is, $L^2(\mathbb{R}) = \{f : \|f\| < \infty\}$. f is *band limited* if $\hat{f}(\omega) = 0$ whenever $|\omega| > \sigma$ for some $\sigma > 0$, where \hat{f} is the *Fourier transform* of $f \in L^2(\mathbb{R})$ defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(t) e^{-i\omega t} dt. \quad (2)$$

In this case, f is also called a σ -band signal. In a simple A/D conversion device, for a σ -band input analog signal f of finite energy, if the sampling interval τ is smaller than the Nyquist sampling interval $\tau_N = \pi/\sigma$, then the discretization in time is reversible by Nyquist sampling theorem [57]. However, the discretization in amplitude leads to an irreversible loss of information. Consequently, the reconstructed analog signal \tilde{f} is generally different from the original signal, and the error $e = f - \tilde{f}$ is referred to as *quantization error*.

In the 1940s, by modeling the quantization error as an uncorrelated uniformly distributed additive noise independent of the input, Bennett [8] proved that the deviation of the additive noise model is given by

$$E|e|^2 = \frac{q^2}{12\tau_N} \tau. \quad (3)$$

This formula suggests that the conversion accuracy can be improved by refining resolution of either discretization in time or discretization in amplitude. Due to the costs involved in building high-resolution quantizers, high accuracy of modern techniques for A/D conversion is achieved by refining time discretization instead. This technique is referred to as an oversampled A/D conversion. The reasons to use an oversampled A/D conversion also include resilience to additive noise [25], resilience to quantization [40], numerical stability of reconstruction [25], greater freedom to capture significant signal characteristics [6], [7], [61], as well as mitigating the effect of losses in packet-based communication systems [38], [39]. But refining the resolution in time only reduces the quantization error by a linear factor while refining that in amplitude reduces the quantization error by a quadratic factor as suggested by (3). This inhomogeneity in time and amplitude dimensions is counterintuitive.

Recently, it was observed that Bennett's estimate is, however, misleading as to the actual accuracy of an oversampled A/D conversion. Moreover, the additive-noise model is not asymptotically valid [10], [11], and experimental results demonstrate that for a high *oversampling ratio* $r = \tau_N/\tau$, the error decay rate of linear reconstruction is in fact lower than that implied by

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W. Chen is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, AB T6G 2V4, Canada (e-mail: wenchen@ece.ualberta.ca).

B. Han and R.-Q. Jia are with the Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada (e-mail: bhan@ualberta.ca; rjia@ualberta.ca).

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the additive noise model [59]. In 1992, Thao and Vetterli [60] showed that the quantization error behaves as

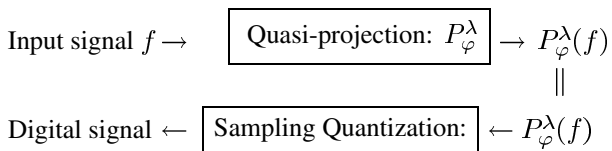
$$\|e\|^2 = O(\tau^2) \quad (4)$$

for the periodic band-limited signals by using nonlinear reconstruction algorithms. Furthermore, the result in [60] has been extended to the setting of the σ -band signals by Cvetkovic and Vetterli [13] in 1999 though the condition is restrictive and rules out all decaying signals.

It is commonly believed that even though oversampling improves accuracy, it has adverse impact on the overall rate-distortion performance of the conversion since the bit rate b increases linearly with the oversampling ratio r , $b = O(r)$, when the standard binary-encoding scheme—pulse-code modulation (PCM) is used. Consequently, $\|e\|^2 = O(\tau^2) = O(1/b^2)$. This poor performance of oversampled A/D conversion appears because the correlation among the quantized samples is neglected in PCM as the sampling interval tends toward zero. Recent research by Cvetkovic and Vetterli [12], [13] reveals that the bit rate required to encode the converted digital signal in a simple oversampled A/D conversion increases only as a logarithm of the oversampling ratio $b = O(\log r)$. This fact follows from the observation that an oversampled A/D conversion amounts to characterizing a signal by its so-called quantization threshold crossings (see Section III). Then $\|e\|^2 = O(\tau^2) = O(2^{-\alpha b})$ for some constant $\alpha \approx 2$. Cvetkovic and Daubechies [14] also proposed an algorithm to demonstrate the theory using *frame theory* [69] and *wavelet theory* [25].

In real-world applications, however, the input analog signals to A/D conversion device are usually non-bandlimited. The conventional approach to treat the non-bandlimited signal is to prefilter the signal by passing it through a low-pass filter [8], [45], [54], [57], [66]. The prefiltered signal is bandlimited and suitable for A/D conversion. The error between the prefiltered signal and the original signal is referred to as *aliasing error*, which can be made arbitrarily small as long as the bandwidth of the filter is large enough.

Recently, the conventional A/D conversion has been extended to the A/D conversion based on shift-invariant spaces $V_\lambda(\varphi)$ [15], [17], in which, the input signal is prefiltered by a *quasi-projection* P_φ^λ into shift invariant space and sampling quantization is performed in the shift-invariant spaces as illustrated by



As consequences of such extension, it offers rich choices to build a nonideal A/D conversion system of high accuracy and low computational complexity, as well as reduces the noise sensitivity and computational complexity in digital-to-analog (D/A) conversion [15], [17]. The accuracy of the extended prefiltering is measured by the aliasing error $f - P_\varphi^\lambda(f)$, which is shown to decay with respect to the dilation λ of the shift-invariant space at the rate as the exponent of the Sobolev space to which the signal f belongs, provided that φ satisfies

the sufficient order Strang–Fix condition [48], [49], [58]. This implies that the aliasing error can be made arbitrarily small as long as the dilation λ , understood to be the bandwidth of the prefilter in the general sense, is large enough. In this paper, we focus on the quantization error of the A/D conversion in shift-invariant spaces. Hence, we will ignore the prefiltering and consider the signal taken from a shift-invariant space.

Following the extended A/D conversion in shift-invariant spaces, it is desirable that the recently derived estimate of quantization error in (4) for the conventional oversampled A/D conversion can be moved to the oversampled A/D conversion in shift-invariant spaces. Our objective in this paper is to derive such advanced estimate of quantization error for the oversampled A/D conversion in shift-invariant spaces, which, not like the conventional case, involves complicated classical analysis.

In this paper, we will introduce a constructive way to establish the estimate $\|e\|^2 = O(\tau^2)$ for the oversampled A/D conversion in shift-invariant spaces, and show that the bit rate required to refine the discretization in time increases only as a logarithm of the oversampling ratio. Therefore, the quantization error of the conversion is an exponentially decaying function of the bit rate. In order to show these results, we should use the sampling theorems for shift-invariant spaces. Therefore, we will recall the advances in sampling in shift-invariant spaces and establish some new sampling theorems for shift-invariant spaces in Section II. With the help of these sampling theorems for shift-invariant spaces, we will be able to introduce a single-bit oversampled A/D conversion scheme in shift-invariant spaces, and estimate the quantization error for the extended oversampled A/D conversion scheme in Section III.

II. SAMPLING IN SHIFT-INVARIANT SPACES

In this section, we will recall the sampling theorems for shift-invariant spaces and establish some new sampling theorems for shift-invariant spaces, which will be used in Section III to establish the estimate of quantization error for the oversampled A/D conversion in shift-invariant spaces.

A. Shift-Invariant Spaces

Let us briefly recall the shift-invariant spaces [27], [46] at first. For a $\lambda \geq 1$, the (*scaled*) shift-invariant space $V_\lambda(\varphi)$ generated by the generator $\varphi \in L^2(\mathbb{R})$ is defined as

$$V_\lambda(\varphi) := \left\{ \sum_{\ell} c_\ell \varphi(\lambda \cdot - \ell) : \sum_{\ell} |c_\ell|^2 < \infty \right\} \subset L^2(\mathbb{R}) \quad (5)$$

where λ , called the *scale* of the $V_\lambda(\varphi)$, is understood to be bandwidth in prefiltering or sampling ratio in sampling. In this paper, we also assume that $\{\varphi(\lambda \cdot - \ell)\}_\ell$ is a Riesz basis¹ of $V_\lambda(\varphi)$, that is, $a \leq G_\varphi \leq b$ almost everywhere for some positive constants a and b [25], [27], where $G_\varphi = \sum_{\ell} |\hat{\varphi}(\cdot + 2\pi\ell)|^2$. It is well

¹A sequence of vectors $\{x_k\}$ in an infinite-dimensional Banach space X is said to be a *Schauder basis* for X if to each vector x in the space there corresponds a unique sequence of scalars $\{c_k\}_k$ such that $x = \sum_{k=-\infty}^{\infty} c_k x_k$. The convergence of the series is understood to be with respect to the strong (norm) topology of X ; in other words, $\|x - \sum_{k=-m}^n c_k x_k\| \rightarrow 0$ as $m, n \rightarrow \infty$. A Schauder basis for a *Hilbert space* is a *Riesz basis* if it is equivalent to an orthonormal basis, that is, if it is obtained from an orthonormal basis by means of a bounded invertible operator (see Young [69]).

known that the Riesz basis $\{\sqrt{\lambda}\varphi(\lambda \cdot -\ell)\}_\ell$ is an orthonormal basis of $V_\lambda(\varphi)$ if and only if $G_\varphi = 1$. In this case, the generator φ and the $V_\lambda(\varphi)$ are said to be orthonormal. Obviously, the Riesz basis $\{\sqrt{\lambda}\varphi(\lambda \cdot -\ell)\}_\ell$ of $V_\lambda(\varphi)$ is regarded to be nearly orthonormal if G_φ is close to 1. We also assume that φ is continuous in this paper. In particular, taking $\text{sinc } t = \frac{\sin \pi t}{\pi t}$, the $V_\lambda(\text{sinc})$ is exactly the $\pi\lambda$ -band signal space of finite energy.

B. Sampling in Band-Limited Shift-Invariant Spaces

In the framework of shift-invariant spaces, the classical sampling theory for band-limited signals can be formulated to sampling in $V_\lambda(\text{sinc})$. A discrete set $X = \{t_k\}_k \subset \mathbb{R}$ is called a *sampling set* for a signal f if f can be perfectly reconstructed from its samples $f(X) = \{f(t_k)\}_k$. If $X = \{k\tau\}_k$ for some constant $\tau > 0$, X is called a *regular sampling set* and the τ is called *sampling interval*. The regular sampling in $V_\lambda(\text{sinc})$ is referred as the Nyquist sampling theorem [57], [66], i.e.,

$$f = \sum_k f(k\tau) \text{sinc} \left(\frac{\cdot}{\tau} - k \right) \quad (6)$$

for any $f \in V_\lambda(\text{sinc})$ and any positive number $\tau \leq 1/\lambda$. The Nyquist sampling interval τ_N is defined by $\tau_N = 1/\lambda$. If $\tau < \tau_N$, X is also called an *oversampling set*.

Let $X = \{\frac{k}{\lambda} + r_k\}_k$ be a perturbation of the regular sampling set $\{\frac{k}{\lambda}\}_k$. Kadec [69] showed that X is a sampling set for $V_\lambda(\text{sinc})$ if $\sup_k |r_k| < \frac{1}{4\lambda}$. Moreover, such sampling set X is *stable*, that is,

$$C^{-1}\|f\| \leq \left(\sum_{t \in X} |f(t)|^2 \right)^{1/2} \leq C\|f\| \quad (7)$$

for some constant $C \geq 1$ and all $f \in V_\lambda(\text{sinc})$.

The general nonuniform sampling in $V_\lambda(\text{sinc})$ is completely understood by the work of Beurling, Landau, and others [9], [34], [50]. The set $X = \{t_k\}_k$ considered by them are *separated*, that is, $d = \inf_{k \neq j} |t_k - t_j| > 0$, where d is the so-called *separation* of X . Let $v^-(r)$ denote the minimum number of points of X to be found in the interval $(-r, r)$, formally

$$v^-(r) = \min_{x \in \mathbb{R}} \#(X \cap (x - r, x + r)).$$

The Beurling lower density is defined by

$$D^-(X) = \liminf_{r \rightarrow \infty} \frac{v^-(r)}{2r}. \quad (8)$$

Then Beurling–Landau theorem says that a separated set X is a stable sampling set for $V_\lambda(\text{sinc})$ if $D^-(X) > \lambda$. Conversely, $D^-(X) \geq \lambda$ if a separated set X is a stable sampling set for $V_\lambda(\text{sinc})$.

C. Advances in Sampling in Shift-Invariant Spaces

Sampling in a general shift-invariant space $V_\lambda(\varphi)$ for a continuous stable generator φ is a relatively recent and active area of research. Observing that sinc is a scaling function of a multiresolution analysis [25] (MRA), Walter [64] first studied the sampling in wavelet subspaces, and found a regular sampling

theorem for a class of wavelet subspaces, which is then extended to the shift sampling in wavelet subspaces by Janssen [43]. On the other hand, Aldroubi and Unser [5] studied the sampling procedure in shift-invariant spaces, and established a more comprehensive sampling theory for shift-invariant spaces.

Suppose that a continuous stable generator φ decays with order $(1 + |t|)^{-s}$ for some $s > 1/2$. Define the function $\hat{\varphi}^*$ by

$$\hat{\varphi}^*(\omega) = \sum_k \varphi(k) e^{-ik\omega}. \quad (9)$$

Chen and Itoh [19] showed that $X = \{\frac{k}{\lambda}\}_k$ is a sampling set for $V_\lambda(\varphi)$ if and only if $1/\hat{\varphi}^* \in L^2[0, 2\pi]$. Moreover, Chen, Itoh, and Shiki [17], [24] showed that $X = \{\frac{k}{\lambda}\}_k$ is a stable sampling set for $V_\lambda(\varphi)$ if and only if

$$0 < \text{ess inf } |\hat{\varphi}^*| \leq \text{ess sup } |\hat{\varphi}^*| < \infty.$$

In this case, there is a so-called *reconstruction frame* $\{S(\lambda \cdot -k)\}_k$ determined by $\hat{S} = \hat{\varphi}/\hat{\varphi}^*$ such that

$$f = \sum_k f\left(\frac{k}{\lambda}\right) S(\lambda \cdot -k), \quad \forall f \in V_\lambda(\varphi). \quad (10)$$

This covers Nyquist sampling theorem since $\widehat{\text{sinc}}^* = 1$. The oversampling $X = \{k2^{-n}\}_k$ in $V_\lambda(\varphi)$ is studied by Walter [65], Xia [67], and Chen and Itoh [20], [21] using MRA[25], which says that

$$f = \sum_k f\left(\frac{k}{2^n \lambda}\right) S(2^n \lambda \cdot -k), \quad \forall f \in V_\lambda(\varphi) \quad (11)$$

provided that φ is refinable, and $\hat{\varphi}^* \neq 0$. However, so far there are very few results on the general oversampling $X = \{k\tau\}_k$ in $V_\lambda(\varphi)$.

For perturbation of regular sampling in $V_\lambda(\varphi)$, $X = \{k + r_k\}_k$, Liu and Walter [53], and Chen, Itoh, and Shiki [22] tried to extend Kadec's theorem to a class of wavelet subspaces. But they actually did not get a Kadec-type extension. Then Chen, Itoh, and Shiki [23] introduced a function class $L_\sigma^\mu[a, b]$ ($\mu > 0$, $\sigma \in [0, 1]$, and $0 \in [a, b] \subset [-1, 1]$), consisting of all the measurable functions, for which the norm

$$\|f\|_{L_\sigma^\mu[a, b]} = \sup_{\{r_k\}_k \subset [a, b]} \frac{\sum_k |f(k + \sigma + r_k) - f(k + \sigma)|}{\sup_k |r_k|^\mu} < \infty. \quad (12)$$

They obtained an estimate for the perturbation $\{r_k\}_k$ such that X is a stable sampling set for $V_\lambda(\varphi)$ if φ is refinable in $L_\sigma^\mu[a, b]$. The result is sharpened in the recent work of Chen, Itoh, and Shiki [24] in the setting of shift-invariant spaces. In particular, $X = \{k + r_k\}_k$ is a stable sampling set for $V_\lambda(\varphi)$ if

$$0 < \text{ess inf } |\hat{\varphi}^*| \leq \text{ess sup } |\hat{\varphi}^*| < \infty$$

$\varphi \in L_0^\mu[a, b]$, and

$$\sup_k |r_k| < \frac{1}{\lambda} \left(\frac{\text{ess inf } |\hat{\varphi}^*|}{\|\varphi\|_{L_0^\mu[a, b]}} \right)^{1/\mu}. \quad (13)$$

For the general nonuniform sampling in a shift-invariant space $V_\lambda(\varphi)$, $X = \{t_k\}_k$, Liu [52], and Aldroubi and Feichtinger [2] applied the Feichtinger–Gröchenig iterative algorithm [30] to spline and shift-invariant spaces, to show that $X = \{t_k\}_k$ is a sampling set if the *maximum gap* $\sup_k |t_{k+1} - t_k|$ between consecutive samples is sufficiently small. Explicit estimates are known only for a few examples and they are far from optimal. In 1999, Aldroubi and Gröchenig [3] succeeded in extending Beurling–Landau’s result to the spline shift-invariant spaces. For the general shift-invariant spaces, even for these generated by a compactly supported function, corresponding results are unknown so far.

D. A Nonuniform Sampling Theorem for Shift-Invariant Spaces

It has been understood that $X = \{t_k\}_k$ is a sampling set for $V_\lambda(\varphi)$ if the maximum gap $\sup_k |t_{k+1} - t_k|$ between consecutive samples is sufficiently small [2]. It will be of great interest to know how small the maximum gap should be. Although this has been discussed in [2], in this subsection, we will establish a nonuniform sampling theorem for a general shift-invariant space, which gives an explicit upper bound for the maximum gap.

For a separated set $X = \{t_k\}_k$, our objective in this subsection is to find some conditions on X such that

$$C^{-1}\|f\| \leq \left(\sum_k |f(t_k)|^2 \right)^{1/2} \leq C\|f\|, \quad f \in V_\lambda(\varphi) \quad (14)$$

for some constant $C \geq 1$; in other words, X is a stable sampling set for $V_\lambda(\varphi)$. We need to use Wirtinger’s inequality [35] described in the following lemma.

Lemma 1: If $f, f' \in L^2(a, b)$, and either $f(a) = 0$ or $f(b) = 0$, then

$$\int_a^b |f(t)|^2 dt \leq \frac{4}{\pi^2} (b-a)^2 \int_a^b |f'(t)|^2 dt. \quad \square$$

With the assistance of Wirtinger’s inequality, we then are able to show the following nonuniform sampling theorem. Our method is inspired by the argument presented in [35].

Theorem 1: Suppose that the generator φ is differentiable and $\sup_\omega G_{\varphi'}(\omega) < \infty$, then the separated set X is a stable sampling set for $V_\lambda(\varphi)$ if the maximum gap δ of X satisfies

$$\delta < \frac{\pi}{\lambda} \inf_\omega \sqrt{\frac{G_\varphi(\omega)}{G_{\varphi'}(\omega)}}. \quad \square$$

Proof: Suppose that the separated set $X = \{t_k\}_k$. Let $s_k = (t_{k-1} + t_k)/2$. Make a piecewise-constant function

$$S_f(t) = \sum_k f(t_k) \chi_{[s_k, s_{k+1})}(t)$$

where $\chi_{[s_k, s_{k+1})}$ is the characteristic function of the interval $[s_k, s_{k+1})$. Since $f(t_k) - S_f(t_k) = 0$, by Wirtinger’s inequality

$$\int_{s_k}^{t_k} |f(t) - S_f(t)|^2 dt \leq \frac{4}{\pi^2} (t_k - s_k)^2 \int_{s_k}^{t_k} |f'(t)|^2 dt$$

$$\text{and} \quad \int_{t_k}^{s_{k+1}} |f(t) - S_f(t)|^2 dt \leq \frac{4}{\pi^2} (s_{k+1} - t_k)^2 \int_{t_k}^{s_{k+1}} |f'(t)|^2 dt.$$

Since $|t_k - s_k| \leq \delta/2$ and $|s_{k+1} - t_k| \leq \delta/2$, we obtain

$$\int_{s_k}^{s_{k+1}} |f(t) - S_f(t)|^2 dt \leq \frac{\delta^2}{\pi^2} \int_{s_k}^{s_{k+1}} |f'(t)|^2 dt$$

that is,

$$\sum_k \int_{s_k}^{s_{k+1}} |f(t) - S_f(t)|^2 dt \leq \sum_k \frac{\delta^2}{\pi^2} \int_{s_k}^{s_{k+1}} |f'(t)|^2 dt.$$

This immediately implies that

$$\|f\| - \|S_f\| \leq \|f - S_f\| \leq \frac{\delta}{\pi} \|f'\|.$$

Then

$$\|f\| - \frac{\delta}{\pi} \|f'\| \leq \|S_f\| \leq \|f\| + \frac{\delta}{\pi} \|f'\|. \quad (15)$$

Suppose that d is the separation of X . Then we derive

$$d \sum_k |f(t_k)|^2 \leq \|S_f\|^2 \leq \delta \sum_k |f(t_k)|^2. \quad (16)$$

On the other hand, since $f = \sum_k c_k \varphi(\lambda \cdot -k)$, we have $f' = \sum_k \lambda c_k \varphi'(\lambda \cdot -k)$. By Parseval identity, we derive

$$\begin{aligned} \|f\|^2 &= \frac{1}{2\pi} \|\hat{f}\|^2 \\ &= \frac{1}{2\pi} \left\| \frac{1}{\lambda} \hat{\varphi} \left(\frac{\cdot}{\lambda} \right) \sum_k c_k e^{-ik\cdot/\lambda} \right\|^2 \\ &= \frac{1}{2\pi\lambda^2} \int_0^{2\pi\lambda} G_\varphi \left(\frac{\omega}{\lambda} \right) \left| \sum_k c_k e^{-ik\omega/\lambda} \right|^2 d\omega. \end{aligned}$$

Similarly, using Parseval identity, we also have

$$\|f'\|^2 = \frac{\lambda^2}{2\pi\lambda^2} \int_0^{2\pi\lambda} G_{\varphi'} \left(\frac{\omega}{\lambda} \right) \left| \sum_k c_k e^{-ik\omega/\lambda} \right|^2 d\omega.$$

Combining these two equations, we derive

$$\begin{aligned} \|f'\|^2 &\leq \frac{\lambda^2}{2\pi\lambda^2} \sup_\omega \left| \frac{G_{\varphi'}(\omega)}{G_\varphi(\omega)} \right| \int_0^{2\pi\lambda} G_\varphi \left(\frac{\omega}{\lambda} \right) \\ &\quad \times \left| \sum_k c_k e^{-ik\omega/\lambda} \right|^2 d\omega \\ &= \lambda^2 \|f\|^2 \sup_\omega \left| \frac{G_{\varphi'}(\omega)}{G_\varphi(\omega)} \right|. \end{aligned}$$

From (15), this immediately implies that

$$\begin{aligned} \left(1 - \frac{\delta\lambda}{\pi} \sqrt{\sup_\omega \left| \frac{G_{\varphi'}(\omega)}{G_\varphi(\omega)} \right|} \right) \|f\| \\ \leq \|S_f\| \leq \left(1 + \frac{\delta\lambda}{\pi} \sqrt{\sup_\omega \left| \frac{G_{\varphi'}(\omega)}{G_\varphi(\omega)} \right|} \right) \|f\|. \end{aligned}$$

From (16), we obtain

$$\begin{aligned} & \frac{\left(1 - \frac{\delta\lambda}{\pi} \sqrt{\sup_{\omega} \left| \frac{G_{\varphi'}(\omega)}{G_{\varphi}(\omega)} \right|}\right)}{\delta} \|f\| \\ & \leq \left(\sum_k |f(t_k)|^2 \right)^{1/2} \\ & \leq \frac{\left(1 + \frac{\delta\lambda}{\pi} \sqrt{\sup_{\omega} \left| \frac{G_{\varphi'}(\omega)}{G_{\varphi}(\omega)} \right|}\right)}{d} \|f\|. \end{aligned} \quad (17)$$

Since $\sup_{\omega} G_{\varphi'}(\omega) < \infty$, (17) shows the right-hand side inequality in (14). In order that (17) shows the left-hand side inequality in (14), it suffices to have

$$1 - \frac{\delta\lambda}{\pi} \sqrt{\sup_{\omega} \left| \frac{G_{\varphi'}(\omega)}{G_{\varphi}(\omega)} \right|} > 0$$

that is equivalent to

$$\delta < \frac{\pi}{\lambda} \inf_{\omega} \sqrt{\frac{G_{\varphi}(\omega)}{G_{\varphi'}(\omega)}}.$$

This completes the proof. \square

If X is a stable sampling set for $V_{\lambda}(\varphi)$, a perfect reconstruction formula using frame theory has been presented in [24]. Since this reconstruction result will be used in Section III to estimate the quantization error, we cite it as follows.

Theorem 2: Assume that φ is a continuous stable generator. If $X = \{t_n\}_n$ is a stable sampling set, then $f = \sum_n f(t_n)S(\lambda \cdot)$ for any $f \in V_{\lambda}(\varphi)$, where the reconstruction frame $\{S_n\}_n$ is derived by the formula

$$S_n = 2\pi \sum_m d_{nm} q_{\varphi}(t_m, \cdot), \quad n \in \mathbb{Z} \quad (18)$$

where $(d_{mn})_{m,n \in \mathbb{Z}}$ is the inverse matrix of the infinite-dimensional matrix $(F_{mn})_{m,n \in \mathbb{Z}}$, and

$$\begin{aligned} F_{nm} &= \int_0^{2\pi} G_{\varphi}^{-1}(\omega) \sum_k \varphi(t_n - k) e^{-ik\omega} \\ & \quad \times \sum_k \varphi(t_m - k) e^{ik\omega} d\omega. \end{aligned} \quad (19)$$

\square

Several remarks are in order.

1) Aldroubi and Gröchenig [3] have derived the right-hand side inequality in (14) under the assumption $\sum_k \sup_{t \in [0,1]} |\varphi(t+k)| < \infty$. But in (17), this condition is removed.

2) Since $\hat{\varphi}'(\omega) = \omega \hat{\varphi}(\omega)$, $G_{\varphi'}$ can be simply calculated by

$$G_{\varphi'}(\omega) = \sum_k |(\omega + k) \hat{\varphi}(\omega + k)|^2.$$

3) In [52], a similar estimate as Theorem 1 has been derived for the iterative algorithm in spline wavelet spaces using $\sup_{\omega} G_{\varphi'}(\omega)$ and $\inf_{\omega} G_{\varphi}(\omega)$. However, the estimate in Theorem 1 is better than that in [52], though the estimate in [52] can be extended to general shift-invariant spaces as done in [18]. As another advantage of Theorem 1, it results in Theorem 2 which is useful in estimating the quantization error in Section III.

E. Numerical Results

We see some typical shift-invariant spaces to demonstrate Theorem 1.

1) *Band-Limited Shift-Invariant Spaces:* In $V_{\lambda}(\text{sinc})$, we have

$$\widehat{\text{sinc}} = \chi_{[-\pi\lambda, \pi\lambda]} \quad \text{and} \quad \widehat{\text{sinc}'} = \pi\lambda \chi_{[-\pi\lambda, \pi\lambda]}.$$

Then

$$\frac{G_{\text{sinc}}}{G_{\text{sinc}'}} = \frac{1}{\pi^2 \lambda^2}.$$

By Theorem 1, this implies that the maximum gap $\delta < 1/\lambda$, which coincides with the conventional Burling–Landau theorem. This also shows that Theorem 1 is optimal for the bandlimited shift-invariant spaces. This optimality has also been indicated in [35].

2) *B-Spline Shift-Invariant Spaces:* In $V_{\lambda}(\beta^N)$, the generator β^N is defined by the N -times convolutions of the characteristic function of the interval $[-1/2, 1/2]$, i.e.,

$$\beta^N := \underbrace{\chi_{[-1/2, 1/2]} * \cdots * \chi_{[-1/2, 1/2]}}_N.$$

Since B-splines are refinable, it satisfies the Strong-Fix condition. Hence, we can perform A/D conversion in $V_{\lambda}(\beta^N)$. Taking the Fourier transform of β^N , we have $\widehat{\beta^N} = \text{sinc}^N(\frac{\cdot}{2\pi})$. This implies that

$$G_{\beta^N} = \sum_k \text{sinc}^{2N}\left(\frac{\cdot}{2\pi} + k\right),$$

and

$$G_{\beta^N} = \sum_k |\cdot + 2\pi k|^2 \text{sinc}^{2N}\left(\frac{\cdot}{2\pi} + k\right).$$

Fig. 1 shows the estimate derived by Theorem 1 versus the the degree N of the B-splines. We will observe that the estimate decrease as N increases. In [3], Aldroubi and Gröchenig showed that the estimate can be $\delta < 1$ for spline shift-invariant spaces. This implies that Theorem 1 is not yet optimal. But the advantage of Theorem 1 is its applicability to general shift-invariant spaces.

3) *Gabor Shift-Invariant Spaces:* The Gaussian kernel is defined by $K(t) = 1/\sqrt{2\pi} e^{-t^2/2}$. Then $\hat{K}(\omega) = e^{-\omega^2/2}$. Make g such that $\hat{g} = \hat{K} \text{sinc}(\frac{\cdot}{2\pi})$. Then g satisfies the Strang–Fix condition and $g = K * \chi_{[-1/2, 1/2]}$. Hence, we can perform A/D conversion in $V_{\lambda}(g)$. Meanwhile, as an advantage of g , we have that g is exponentially decaying in both time and frequency domains. Notice that

$$G_g(\omega) = \sum_k \left| e^{(\omega + 2\pi k)^2} \text{sinc}^2\left(\frac{\omega}{2\pi} + k\right) \right|^2$$

and

$$G_{g'}(\omega) = \sum_k \left| (\omega + 2\pi k)^2 e^{(\omega + 2\pi k)^2} \text{sinc}^2\left(\frac{\omega}{2\pi} + k\right) \right|^2.$$

Matlab shows that the estimate is $\delta < 0.9/\lambda$.

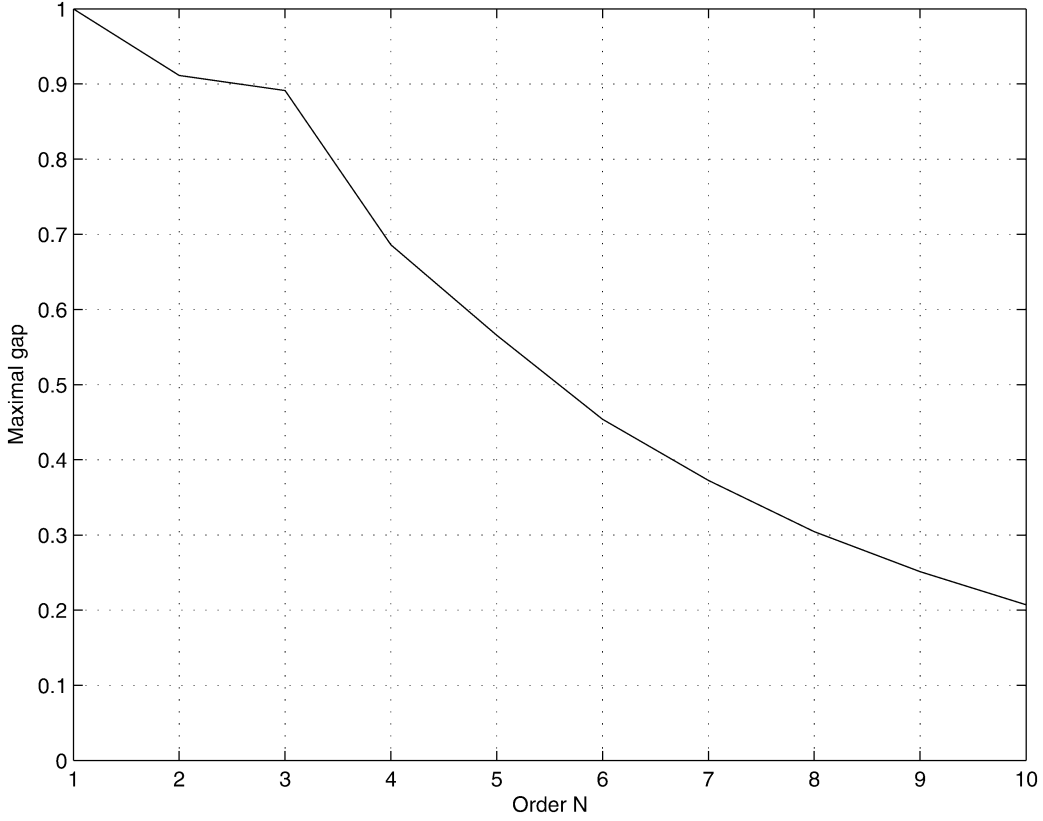


Fig. 1. The upper bound versus the order N of the B-spline.

III. QUANTIZATION IN SHIFT-INVARIANT SPACES

We are ready to introduce a dithered single-bit oversampled A/D conversion scheme in shift-invariant spaces using the nonuniform sampling theorem established in the last section. Our objective in this section is to establish the estimate of quantization error $|e|^2 = O(\tau^2)$ for the oversampled A/D conversion in shift-invariant spaces. We will also demonstrate the fact that the bit rate required to encode the converted signal only increases as logarithm of the sampling rate. Therefore, the quantization error is an exponentially decaying function of the bit rate.

A. A Single-Bit Oversampled A/D Conversion Scheme in Shift-Invariant Spaces

We now introduce the dithered single-bit oversampled A/D conversion scheme [14] for shift-invariant spaces. In this scheme, we consider such a signal $f \in V_\lambda(\varphi)$ which is bounded above by a finite number $b > 0$, that is,

$$f \in V_\lambda^b(\varphi) := V_\lambda(\varphi) \cap \{f \mid \|f\|_\infty < b\}.$$

In real-world applications, this is exactly what one does in practice (*gain control*) [28]. Without loss of generality, we assume $b = 1$ for convenience. A so-called deterministic *dither function* h plays a crucial role in this scheme (see Fig. 2). Usually, the dither function h is differentiable and satisfies

$$\begin{cases} h' \in L^\infty(\mathbb{R}) \\ \left| h\left(\frac{n}{\mu}\right) \right| \geq \gamma > 1 \\ \text{sign } h\left(\frac{n}{\mu}\right) = -\text{sign } h\left(\frac{n+1}{\mu}\right). \end{cases}$$

An appropriate example of a dither function is the sinusoidal function $h = \gamma \sin \mu\pi(\cdot)$.

Let $I_n = (n/\mu, (n+1)/\mu)$. Since $|(f-h)(n/\mu)|$ is bounded below by $\gamma - 1 > 0$, and alternates in sign consecutively, there is at least one zero crossing $t_n \in I_n$ such that $(f-h)(t_n) = 0$. Since

$$\begin{aligned} 0 < \gamma - 1 &\leq |(f-h)(k/\mu)| \\ &= |(f-h)(k/\mu) - (f-h)(t_n)| \\ &\leq \|f' - h'\|_\infty |k/\mu - t_n| \end{aligned} \quad (20)$$

we have

$$\begin{aligned} |t_{n+1} - t_n| &= |t_{n+1} - (n+1)/\mu| + |(n+1)/\mu - t_n| \\ &\geq \frac{2(\gamma-1)}{\|f' - h'\|_\infty}. \end{aligned} \quad (21)$$

It shows that $X = \{t_n\}_n$ is separated. Since $|t_n - t_{n+1}| \leq 2/\mu$, by Theorem 1, $X = \{t_n\}_n$ forms a stable sampling set for $V_\lambda(\varphi)$ if

$$\mu > \frac{\lambda}{2\pi} \sup_{\omega} \sqrt{\frac{G_{\varphi'}(\omega)}{G_{\varphi}(\omega)}}.$$

Let τ be a sufficient small positive number such that $1/(\tau\mu)$ is a positive integer. Let $N_\mu = 1/(\tau\mu)$. Then

$$I_n = n/\mu + \cup_{m=0}^{N_\mu-1} [\tau m, \tau(m+1)].$$

For $m = 0, \dots, N_\mu - 1$, if

$$(f-h)(n/\mu + \tau m) \times (f-h)(n/\mu + \tau(m+1)) \leq 0$$

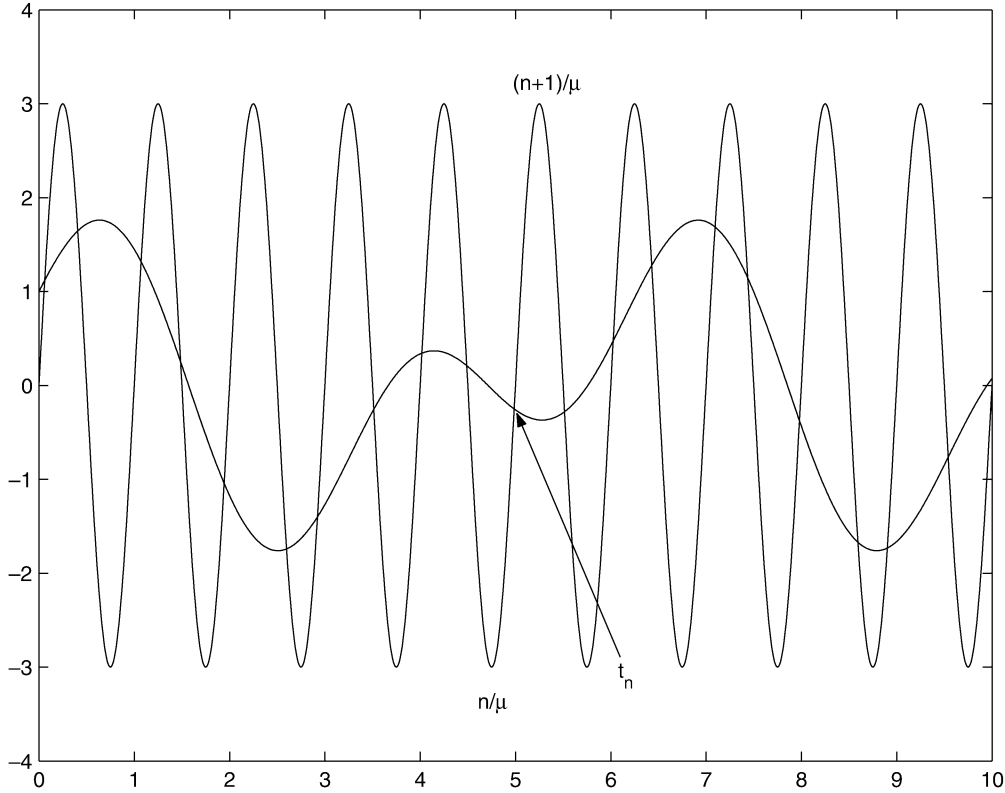


Fig. 2. A dithered A/D conversion scheme $C_{\tau,\mu}^h$. h is a dither function. f is an input analog signal to the A/D conversion. The functions h and f have at least one intersection t_n in each interval $I_n = (n/\mu, (n+1)/\mu)$. I_n is separated by the sampling interval τ as $I_n = n/\mu + \cup_0^{N_\mu} [\tau m, \tau(m+1)]$. The dithered A/D conversion only records the first m_n such that t_{m_n} is an intersection in I_n . Then the A/D conversion approximates $f(t_{m_n})$ by $h(s_{m_n})$ with $s_{m_n} = t_{m_n} + \tau/2$. By Theorem 1, $\{t_{m_n}\}_n$ forms a stable sampling set for $V(\varphi)$ if μ is large enough.

then there is a zero crossing $t_{m,n} \in [\tau m, \tau(m+1)] + n/\mu$. Let $t_{m_n} = \min_m t_{m,n}$, and $s_{m_n} = n/\mu + \tau m_n + \tau/2$. When τ is small enough, $t_{m_n} \approx s_{m_n}$. Then $h(s_{m_n}) \approx h(t_{m_n}) = f(t_{m_n})$. It suggests that we can define an A/D converter $C_{\tau,\mu}^h : V_\lambda^b(\varphi) \rightarrow l^2(\mathbb{Z})$ as

$$C_{\tau,\mu}^h(f) = \{h(s_{m_n})\}_n. \quad (22)$$

The bits needed to specify m_n are $\log_2 N_\mu = -\log_2(\tau\mu)$. Specifying the information about zero crossing of $f - h$ on the interval $[-n/\mu, n/\mu]$ requires $-2n \log_2(\tau\mu)$ bits. Thus, the bit rate needed for specifying the location of one zero crossing within I_n is $R = -\mu \log_2(\tau\mu)$.

In order to find the zero crossing t_{m_n} within I_n , we only need to look at $\text{sign}(f - h)(\tau m)$. The first m from $m = 0$ to $N_\mu - 1$ such that $\text{sign}(f - h)(n/\mu + \tau(m+1))$ changes sign is the m_n . But we only need one bit to record the $\text{sign}(f - h)(\tau m)$. This is why the dithered A/D conversion scheme is called a *single-bit scheme*.

B. Accuracy Analysis for Quantization Error

For a given dither function h , an analog signal can be converted into a digital signal by the A/D converter $C_{\tau,\mu}^h$. In this subsection, we discuss how much information about a signal $f \in V_\lambda^b(\varphi)$ can be reconstructed from the converted digital signal $C_{\tau,\mu}^h(f)$. We need the weighted Wiener amalgam space

W_r^1 for some $r > 0$, consisting of all the measurable functions φ , for which the norm

$$\|\varphi\|_{W_r^1} = \|(1 + |\cdot|)^r \varphi\|_{W^1} < \infty \quad (23)$$

where the Wiener amalgam space W^1 consisting of all the measurable functions φ , for which the norm

$$\|\varphi\|_{W^1} = \sum_k \sup_{t \in [0,1]} |\varphi(t+k)| < \infty.$$

We also need the following proposition about shift-invariant spaces.

Proposition 1: Assume that φ is a differentiable stable generator in W_r^1 such that $\varphi' \in W^1$. Then there is an orthonormal differentiable stable generator ψ in W_r^1 such that $\psi' \in W^1$ and $V_\lambda(\psi) = V_\lambda(\varphi)$. \square

In order to show this proposition, we need the following extension of Wiener theorem on the Fourier coefficients of a 2π -periodic function [51].

Lemma 2: Suppose $g(\omega) = \sum_k a_k e^{ik\omega} \neq 0$ for all $\omega \in \mathbb{R}$ with $\{(1 + |k|)^r a_k\}_k \in l^1$. If P is complex valued and holomorphic on some open set containing $g(\mathbb{R})$, and $P(g(\omega)) = \sum_k b_k e^{-ik\omega}$, then $\{(1 + |k|)^r b_k\}_k \in l^1$. \square

Proof of Proposition 1: We consider $V_1(\varphi)$ and shift the result to $V_\lambda(\varphi)$ by rescaling. Take $\psi \in V_1(\varphi)$ such that

$$\hat{\psi} = \hat{\varphi} / \sqrt{G_\varphi}. \quad (24)$$

Then ψ is an orthonormal stable generator of $V_1(\varphi)$ [25], [27]. Note that the 2π -periodic function $G_\varphi(\omega) = \sum_k g_k e^{-ik\omega}$ with the Fourier coefficients

$$g_k = \int_{\mathbb{R}} \varphi(t) \varphi(t+k) dt. \quad (25)$$

Then

$$\begin{aligned} & \sum_k (1+|k|)^r |g_k| \\ & \leq \int_{\mathbb{R}} |\varphi(t)| \sum_k (1+|k|)^r |\varphi(t+k)| dt \\ & \leq \int_{\mathbb{R}} |\varphi(t)| \sum_k (1+|t+k|+|t|)^r |\varphi(t+k)| dt \\ & \leq \int_{\mathbb{R}} |\varphi(t)| (1+|t|)^r \sum_k (1+|t+k|)^r |\varphi(t+k)| dt \\ & \leq \|(1+|\cdot|)^r \varphi\|_1 \|\varphi\|_{W_r^1} \\ & \leq \|\varphi\|_{W_r^1}^2 < \infty. \end{aligned} \quad (26)$$

Write $1/\sqrt{G_\varphi} = \sum_k h_k e^{-ik\omega}$. Then $\psi = \sum_k h_k \varphi(\cdot - k)$. By Lemma 2, we have $\sum_k (1+|k|)^r |h_k| < \infty$ since $G_\varphi > 0$. Therefore,

$$\begin{aligned} & \|(1+|\cdot|)^r \psi\|_{W^1} \\ & \leq \left\| (1+|\cdot|)^r \sum_k |h_k| \varphi(\cdot - k) \right\|_{W^1} \\ & \leq \left\| \sum_k |h_k| (1+|k|)^r (1+|\cdot-k|)^r |\varphi(\cdot-k)| \right\|_{W^1} \\ & \leq \sum_k |h_k| (1+|k|)^r \|(1+|\cdot-k|)^r |\varphi(\cdot-k)|\|_{W^1} \\ & = \|\varphi\|_{W_r^1} \sum_k (1+|k|)^r |h_k| < \infty. \end{aligned} \quad (27)$$

It shows $\psi \in W_r^1$. On the other hand

$$\begin{aligned} \sum_l \sup_{t \in [0,1]} |\psi'(t+l)| & \leq \sum_k |h_k| \times \sum_l \sup_{t \in [0,1]} |\varphi'(t+l-k)| \\ & = \|\varphi'\|_{W^1} \sum_k |h_k|. \end{aligned} \quad (28)$$

We therefore conclude that $\psi' \in W^1$. This completes the proof. \square

We also need another result about the decaying property of the entries of an infinite-dimensional matrix found by Jaffard [45]. This result has been used for sampling in shift-invariant spaces by Gröchenig in [36].

Lemma 3: Suppose that the infinite-dimensional matrix $A = (a_{jk})$ is invertible and bounded as an operator: $l^2 \rightarrow l^2$. Let $A^{-1} = (b_{jk})$. If $|a_{jk}| = O(|j-k|^{-1-r})$ for some $r > 0$, then $|b_{jk}| = O(|j-k|^{-1-r})$. \square

Now we are ready to estimate the quantization error for the dithered A/D conversion scheme presented in the last subsection. It is formulated in the following theorem.

Theorem 3: Assume that φ is a stable generator in W_r^1 such that $\varphi' \in W^1$. Suppose that the differentiable dither function h satisfies $h' \in L^\infty(\mathbb{R})$. Then the quantization error e in the dithered A/D conversion scheme $C_{\tau,\mu}^h$ behaves as $|e|^2 = O(\tau^2)$. \square

Proof: By Proposition 1, there is an orthonormal differentiable stable generator $\psi \in W_r^1$ such that $\psi' \in W^1$ and $V_\lambda(\psi) = V_\lambda(\varphi)$. Without loss of generality, we can assume that φ is an orthonormal stable generator. Then $G_\varphi = 1$. Let \tilde{f} be the reconstructed analog signal from the digital signal $C_{\tau,\mu}^h(f) = \{h(s_{m_n})\}_n$ by the reconstruction formula in Theorem 2. Then $\tilde{f} = \sum_n h(s_{m_n}) S_n(\lambda \cdot)$ and

$$S_n = 2\pi \sum_m h_{nm} q_\varphi(t_m, \cdot) \quad (29)$$

where $(h_{nm})_{n,m \in \mathbb{Z}}$ is the inverse matrix of the infinite-dimensional matrix $F = (F_{mn})_{m,n \in \mathbb{Z}}$, and

$$\begin{aligned} F_{mn} &= \int_0^{2\pi} G_\varphi^{-1}(\omega) \sum_k \varphi(t_n - k) e^{-ik\omega} \\ & \quad \times \sum_k \varphi(t_m - k) e^{ik\omega} d\omega \\ &= \int_0^{2\pi} \sum_k \varphi(t_n - k) e^{-ik\omega} \sum_k \varphi(t_m - k) e^{ik\omega} d\omega \\ &= 2\pi \sum_k \varphi(t_n - k) \varphi(t_m - k). \end{aligned} \quad (30)$$

The assumption $\varphi \in W_r^1$ implies that

$$|\varphi(t_n - k)| \leq c(1+|t_n - k|)^{-1-r}$$

for some constant $c > 0$. Hence,

$$\begin{aligned} |F_{mn}| & \leq 2\pi c^2 \sum_k (1+|t_n - k|)^{-1-r} (1+|t_m - k|)^{-1-r} \\ &= 2\pi c^2 \left(\sum_{|t_n - t_m| \geq 2|t_n - k|} + \sum_{|t_n - t_m| < 2|t_n - k|} \right) \\ & \quad \times [(1+|t_n - k|)(1+|t_m - k|)]^{-1-r} \\ & \leq 2\pi c^2 \sum_{|t_n - t_m| \geq 2|t_n - k|} [(1+|t_n - k|) \\ & \quad \times (1+|t_n - t_m| - |t_n - k|)]^{-1-r} \\ & \quad + 2\pi c^2 \sum_{|t_n - t_m| < 2|t_n - k|} [(1+|t_n - t_m|/2) \\ & \quad \times (1+|t_m - k|)]^{-1-r}. \end{aligned} \quad (31)$$

Simplifying the preceding inequality, we have

$$\begin{aligned} |F_{mn}| & \leq 2\pi c^2 (1+|t_n - t_m|/2)^{-1-r} \\ & \quad \times \sum_{|t_n - t_m| \geq 2|t_n - k|} (1+|t_n - k|)^{-1-r} \\ & \quad + 2\pi c^2 (1+|t_n - t_m|/2)^{-1-r} \\ & \quad \times \sum_{|t_n - t_m| < 2|t_n - k|} (1+|t_m - k|)^{-1-r}. \end{aligned} \quad (32)$$

This finally implies that

$$\begin{aligned} |F_{mn}| & \leq \pi c^2 (1+|t_n - t_m|)^{-1-r} \\ & \quad \times \sum_k [(1+|t_n - k|)^{-1-r} + (1+|t_m - k|)^{-1-r}] \\ & \leq C(1+|t_n - t_m|)^{-1-r} \end{aligned} \quad (33)$$

with

$$C = 2\pi c^2 \sup_{t \in [0,1]} \sum_k (1+|t - k|)^{-1-r} < \infty.$$

As we have proved in the previous subsection in (21), we have

$$|t_n - t_{n+1}| \geq \frac{2(\gamma - 1)}{\|f' - h'\|_\infty} =: d_1. \quad (34)$$

Therefore, $|t_n - t_m| \geq d_1|m - n|$ and consequently

$$|F_{mn}| = O((1 + |m - n|)^{-1-r}).$$

By Lemma 3, we derive $|d_{mn}| = O((1 + |m - n|)^{-1-r})$. Then

$$\begin{aligned} |e(t)| &= |f(t) - \tilde{f}(t)| \\ &= \left| \sum_n [h(t_n) - h(s_{m_n})] S_n(t) \right| \\ &\leq \|h'\|_\infty \left(\frac{\tau}{2} \right) \left| \sum_{n,m} d_{nm} \sum_k \varphi(t_m - k) \varphi(t - k) \right| \\ &\leq \|h'\|_\infty \left(\frac{\tau}{2} \right) \sum_k |\varphi(t - k)| \sum_m |\varphi(t_m - k)| \sum_n |d_{nm}| \\ &\leq (d_1^{-1} + 1) \|\varphi\|_{W^1}^2 \|h'\|_\infty \left(\frac{\tau}{2} \right) \sum_k (1 + |k|)^{-1-r} \\ &= O(\tau). \end{aligned} \quad (35)$$

So, we conclude that $|e|^2 = O(\tau^2)$. This completes the proof. \square

Since the bit rate R required to encode the converted digital signal only increases as a logarithm of the sampling ratio τ , $R = \mu \log_2(\tau \mu)$. Therefore the quantization error is an exponentially decaying function of the bit rate $|e| = O(2^{-R/\mu})$. Since only one bit is required for quantization, the dithered A/D conversion scheme is easy to be implemented in practice.

Recently, as a promising oversampled A/D conversion scheme, the sigma-delta modulation [32], [33] has attained tremendous attention. Following Gray's accuracy analysis for the sigma-delta modulator with the dc and sinusoidal inputs, Gunturk [41], Daubechies and Devore [26], and the present authors [16] recently considered the sigma-delta modulation with the input of bandlimited signals. As a future work, it is also of great interest to extend this result to the shift-invariant spaces.

IV. CONCLUSION

In this paper, we study the oversampled A/D conversion in shift-invariant space, in which an analog signal is prefiltered by a quasi-projection into the shift-invariant spaces, and sampling quantization is performed in the shift-invariant spaces. It has been shown that the aliasing error between the prefiltered signal and the original signal can be made arbitrarily small as long as the dilation of the shift-invariant space is large enough. By ignoring the aliasing error in this paper, the accuracy of the extended A/D conversion is then determined by the quantization error. In this paper, we introduce a constructive method to establish the estimate of quantization error, and successfully establish the estimate $|e|^2 = O(\tau^2)$ with respect to the sampling interval τ . Meanwhile, the bit rate required to encode the converted signal only increases as logarithm of the bit rate. Therefore, the quantization error is an exponentially decaying function of the bit rate. The theory and methodology in this paper will stimulate the engineering research in A/D conversion in

shift-invariant spaces and provide a possible suggestion for the development of the next-generation communication systems.

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