PAPER

Oversampling Theorem for Wavelet Subspace

Wen CHEN†, ***Nonmember and*** **Shuichi ITOH**†, ***Member***

SUMMARY An oversampling theorem for regular sampling in wavelet subspaces is established. The sufficient-necessary condition for which it holds is found. Meanwhile the truncation error and aliasing error are estimated respectively when the theorem is applied to reconstruct discretely sampled signals. Finally an algorithm is formulated and an example is calculated to show the algorithm.
key words: *sampling, oversampling, scaling function, wavelet subspaces, truncation error, aliasing error*

1. Introduction and Preliminaries

Sampling theorem is a classical problem which studies how to reconstruct the original signals from their discrete samples. For a finite energy band-limited signal $f(t)$, i.e., $f(t) \in L^2(R)$ and $\operatorname{supp} \hat{f}(\cdot) \subset [-\sigma, \sigma]$, the famous Shannon sampling theorem gives the recovering formula

$$f(t) = \sum_n f(n\pi/\sigma) \frac{\sin \sigma(t - n\pi/\sigma)}{\sigma(t - n\pi/\sigma)}, \tag{1}$$

where $\hat{f}(\omega)$ is the Fourier transform of $f(t)$ defined by $\hat{f}(\omega) = \int_R f(t) e^{-it\omega} dt$, and $\operatorname{supp} \hat{f}(\cdot)$ is the support set of $\hat{f}(\omega)$ defined by $\operatorname{supp} \hat{f}(\cdot) = \{\omega : \hat{f}(\omega) \neq 0\}$. If we let $\sigma = 2^m \pi$, $m \in Z$, Shannon sampling theorem can be in fact regarded as a special case of sampling in a wavelet subspace with $\varphi(t) = \sin \pi t / \pi t$ playing the role of scaling function of MRA $\{V_m = \overline{\operatorname{span}}\{\varphi(2^m t - n)\}_n\}_m$. Realizing this property, Walter [13] established a recovering formula for the sampling in a class of wavelet subspaces.

Suppose $\varphi(t)$ is a continuous scaling function of an MRA $\{V_m\}_{m \in Z}$ such that $|\varphi(t)| \leq O(|t|^{-1-\varepsilon})$ for some $\varepsilon > 0$ when $|t| \to \infty$. Let $\hat{\varphi}^*(\omega) = \sum_k \hat{\varphi}(\omega + 2k\pi)$. Walter showed that, in orthonormal case, if $\hat{\varphi}^*(\omega) \neq 0$ there is an $S(t) \in V_0$ such that

$$f(t) = \sum_{n \in Z} f(n) S(t - n) \quad \text{for } f \in V_0 \tag{2}$$

holds. Following Walter [13]'s work, Janssen [9] studied the shift sampling case by using Zak-transform. Xia-Zhang [17] discussed the so-called sampling property, i.e.,

$$f(t) = \sum_n f(n) \varphi(t - n) \quad \text{for } f \in V_0. \tag{3}$$

It is also shown that $\varphi(t)$ satisfies sampling property if and only if $\hat{\varphi}^*(\omega) = 1$ holds for *a.e.* $\omega \in R$. Obviously the constraint is too strong to include many important scaling functions, such as Meyer scaling function and certain of Daubechies scaling functions. Thus Walter [14] proposed a weaker expression as

$$f(t) = \sum_n f\left(\frac{n}{2}\right) \varphi(2t - n) \quad \text{for } f \in V_0. \tag{4}$$

Xia [16] extended (4) to a more general form as that for some $J \in Z^+ \cup \{0\}$,

$$f(t) = \sum_n f\left(\frac{n}{2^J}\right) \varphi(2^J t - n) \quad \text{for } f \in V_0. \tag{5}$$

(5) is the so-called oversampling property with rate J. It is also shown that $\varphi(t)$ satisfies oversampling property with rate J ($J \in Z^+ \cup \{0\}$) if and only if $\hat{\varphi}(\omega) = \hat{\varphi}_J^*(\omega) \hat{\varphi}(2^{-J}\omega)$ holds for *a.e.* $\omega \in R$, where $\hat{\varphi}_J^*(\omega) = \sum_n \hat{\varphi}(\omega + 2^{J+1} n\pi)$. Oversampling property does cover many important scaling functions and it was shown by Chen-Itoh [6] that all bounded interval band orthonormal scaling functions show oversampling property with rate J for $J \in Z^+$.

However all these works on sampling in wavelet subspaces at least have some of the following three shortcomings.

1. The continuity constraint imposed on scaling function, which even excludes Haar scaling function (Walter [13] wrongly applied his theorem to calculate it).
2. The decay constraint imposed on scaling function ($|\varphi(t)| \leq O(|t|^{-1-\varepsilon})$ for some $\varepsilon > 0$ when $|t| \to \infty$), which even excludes Shannon scaling function.
3. The constraint $\hat{\varphi}(\omega) = \hat{\varphi}_J^*(\omega) \hat{\varphi}(2^{-J}\omega)$ is too restrictive to include a broad class of important scaling functions (even some bounded interval band scaling functions).

Recently Chen-Itoh [5] find the fact that there is an $S(t) \in V_0$ such that (2) holds in $L^2(R)$-sense if and only

Manuscript received May 14, 1997.
Manuscript revised September 25, 1997.
†The authors are with the Department of Information Network Science, Graduate School of Information Systems, University of Electro-Communications, Chofu-shi, 182 Japan.

if $\frac{1}{\hat{\varphi}^*(\omega)} \in L^2[-\pi,\pi]$ holds. Although Chen-Itoh [5]'s work removes the continuity and decay constraints imposed on scaling functions and obtain a sufficient-necessary condition for (2) to hold, It can not cover the oversampling property case ($\hat{\varphi}(\omega) = \hat{\varphi}_J^*(\omega)\hat{\varphi}(2^{-J}\omega)$ for any $J \in Z^+$). Meanwhile there are many scaling functions neither show oversampling property are nor captured by Chen-Itoh [5]'s Theorem. For example, take $\varphi_c(t)$ such that

$$\hat{\varphi}_c(\omega) = \begin{cases} v(\omega), & |\omega| \leqq \frac{2\pi}{3}, \\ 1, & \frac{2\pi}{3} < |\omega| \leqq \pi, \\ -1, & \pi < |\omega| \leqq \frac{4\pi}{3}, \\ 0, & \frac{4\pi}{3} < |\omega|, \end{cases} \tag{6}$$

where $1 \leqq |v(\omega)| \leqq 2$. It is easy to show that $\varphi_c(t)$ is a scaling function (refer to Boor-Devore-Ron [3]). But $\hat{\varphi}_c^*(\omega) = 0$ on $[-\pi, -\frac{2\pi}{3}] \cup [\frac{2\pi}{3}, \pi]$. Obviously $\frac{1}{\hat{\varphi}_c^*(\omega)} \in L^2[-\pi,\pi]$ does not hold. It can also be shown that $\varphi_c(t)$ does not show oversampling property with rate J for any $J \in Z^+$ except for $v(\omega) \equiv 1$ (refer to Xia [16] and Chen-Itoh [6]). Therefore we can neither apply the Sampling Theorem (see Walter [13] and Chen-Itoh [5]) nor use the oversampling property (see Walter [14], Xia [16] and Chen-Itoh [6]) to recover signals by $\varphi_c(t)$. Our task in this paper is to establish the so-called oversampling theorem for sampling in wavelet subspaces, which can capture these scaling functions like the above $\varphi_c(t)$. Since truncation error and aliasing error should be estimated when recovering signals by using scaling functions, we present two methods to estimate them respectively.

Let us now introduce MRA (Multi Resolution Analysis) which has been aforementioned. For more details see Long-Chen [10] or Long-Chen-Yuan [11] or any books on wavelets such as Chui [7] and Meyer [12]. A so-called MRA $\{V_m\}_{m\in Z}$ is a family of subspaces of $L^2(R)$, which satisfies (1) $V_m \subset V_{m+1}$, $\overline{\cup_m V_m} = L^2(R)$, and $\cap_m V_m = \{\theta\}$; (2) $f(t) \in V_m$ if and only $f(2t) \in V_{m+1}$; (3) There exists a function $\varphi(t) \in V_0$ (called scaling function) such that $\{\varphi(t-n)\}_n$ forms a Riesz basis of V_0. Each V_m is called to be a wavelet subspace. A scaling function $\varphi(t)$ is said to be orthogonal (resp. orthonormal) if $\{\varphi(x-n)\}_n$ forms an orthogonal (resp. orthonormal) basis of V_0. Let $\varphi(t)$ be a scaling function of MRA $\{V_m\}_m$. Then $\{\varphi(2t-n)\}_{n\in Z}$ is a Riesz basis of V_1. Therefore there is a sequence $\{c_k\}_k \in l^2$ such that $\varphi(t) = \sum_k c_k\varphi(2t-k)$. Let $W_0 = V_1 \ominus V_0$ be the direct complement of V_0 in V_1, and $\psi(t) \in W_0$. Then there is a sequence $\{d_k\}_k \in l^2$ such that $\psi(t) = \sum_k d_k\varphi(2t-k)$. If $\{\psi(t-k)\}_k$ is a Riesz basis of W_0, $\psi(t)$ is said to be the wavelet of MRA$\{V_m\}_m$. Therefore for $f(t) \in V_{m+1} = W_m \bigoplus V_m$, there must be $\{a_n\}_n \in l^2$ and $\{b_n\}_n \in l^2$ such that

$$f(t) = \sum_n a_n\varphi(2^m t - n) + \sum_n b_n\psi(2^m t - n). \tag{7}$$

$\{b_n\}_n$ is called to be the wavelet coefficients of $f(t)$ in W_m. The above argument also implies that

$$\hat{\varphi}(\omega) = m_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right), \tag{8}$$

$$\hat{\psi}(\omega) = m_1\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) \tag{9}$$

hold for *a.e.* $\omega \in R$, where $m_0(\omega) = \frac{1}{2}\sum_k c_k e^{-ik\omega}$ and $m_1(\omega) = \frac{1}{2}\sum_k d_k e^{-ik\omega}$. Take $G_\varphi(\omega) = (\sum_n |\hat{\varphi}(\omega + 2n\pi)|^2)^{1/2}$. It is well known (see Chui [7], Meyer [12] or Walter [15]) that

$$0 < \|G_\varphi(\omega)\|_0 \leqq \|G_\varphi(\omega)\|_\infty < \infty \tag{10}$$

always holds, and $\varphi(t)$ is orthonormal if and only if $G_\varphi(\omega) = 1$ holds for *a.e.* $\omega \in R$,

Finally we introduce some notations used in this paper. For measurable subset $E \subset R$, $|E|$ denotes the measure of E. For measurable function $f(t)$, we write

$$\|f(t)\| = \left(\int_R |f(t)|^2 dt\right)^{1/2}, \tag{11}$$

$$\|f(t)\|_0 = \sup_{|E|=0} \inf_{R\ominus E} |f(t)|, \tag{12}$$

$$\|f(t)\|_\infty = \inf_{|E|=0} \sup_{R\ominus E} |f(t)|, \tag{13}$$

$$\chi_E(t) = \begin{cases} 1 & t \in E \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where $\chi_E(t)$ is called the characteristic function of set E.

2. Oversampling Theorem

We consider the $\varphi_c(t)$ defined in the Introduction. Although $\frac{1}{\hat{\varphi}_c^*(\omega)} \in L^2[-\pi,\pi]$ does not hold, we find $\hat{\varphi}_c^*(\omega) = v(\omega)$ on $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$. It is exactly the support set of $m_0(\omega)$ on $[-\pi,\pi]$ ($\operatorname{supp}\hat{\varphi}_c(2\cdot) = [-\frac{2\pi}{3}, \frac{2\pi}{3}]$, $\operatorname{supp}\hat{\varphi}_c(\cdot) = [-\frac{4\pi}{3}, \frac{4\pi}{3}]$ and $\hat{\varphi}_c(2\omega) = m_0(\omega)\varphi_c(\omega)$ force that $\operatorname{supp} m_0(\cdot)|_{[-\pi,\pi]} = [-\frac{2\pi}{3}, \frac{2\pi}{3}]$ holds. In fact $m_0(\omega)|_{[-\pi,\pi]} = \frac{\hat{\varphi}(2\omega)}{v(\omega)}$ holds in this case.). This implies that we can consider $\frac{1}{\hat{\varphi}_c^*(\omega)}\chi_{\operatorname{supp} m_0(\cdot)}(\omega)$ instead of $\frac{1}{\hat{\varphi}_c^*(\omega)}$. Fortunately it does work.

Theorem 1: Let $\varphi(t)$ be the scaling function of MRA$\{V_m\}_m$ such that $\{\varphi(n)\}_n \in l^2$. Then there is an $S(t) \in V_0$ and a $J \in Z^+$ such that

$$f(t) = \sum_n f(2^{-J}n)S(2^J t - n) \quad \text{for } f(t) \in V_0 \tag{15}$$

holds in $L^2(R)$-sense if and only if

$$\frac{1}{\hat{\varphi}^*(\omega)}\chi_{\cap_{j=0}^{J-1} 2^{-j}\operatorname{supp} m_0(\cdot)}(\omega) \in L^2[-\pi,\pi] \tag{16}$$

holds. In this case $\hat{S}(\omega) = \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)}$ holds for *a.e.* $\omega \in \cap_{j=0}^{J-1} 2^{-j} \mathrm{supp}\, m_0(\cdot)$.

Proof Step 1: sufficiency.

Assume $\frac{1}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp}\, m_0(\cdot)}(\omega) \in L^2[-\pi, \pi]$. Then $\hat{\varphi}^*(\omega) \neq 0$ holds for *a.e.* $\omega \in \cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\omega)$, and there is a $\{c_k\}_k \in l^2$ such that

$$\frac{1}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\cdot)}(\omega) = \sum_k c_k e^{ik\omega} \tag{17}$$

holds in $L^2[-\pi, \pi]$-sense. We now let

$$F(\omega) = \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\cdot)}(\omega). \tag{18}$$

Then

$$\int_R |F(\omega)|^2 d\omega$$

$$= \int_R \left| \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\cdot)}(\omega) \right|^2 d\omega \tag{19}$$

$$= \int_{-\pi}^{\pi} \frac{\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2}{|\hat{\varphi}^*(\omega)|^2} \cdot \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\cdot)}(\omega) d\omega \tag{20}$$

$$\leq \|G_\varphi(\omega)\|_\infty^2 \int_{-\pi}^{\pi} \frac{1}{|\hat{\varphi}^*(\omega)|^2} \cdot \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\cdot)}(\omega) d\omega \tag{21}$$

It is easy to see $F(\omega) \in L^2(R)$ due to (10) and (21). Hence we can take the inverse Fourier transform of $F(\omega)$ in $L^2(R)$ denoted by $S(t)$ (refer to Introduction), i.e., we can derive

$$\hat{S}(\omega) = \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\cdot)}(\omega) \tag{22}$$

or

$$\hat{\varphi}(\omega) \chi_{\cap_{j=0}^{J-1} 2^{-j} \mathrm{supp} m_0(\cdot)}(\omega) = \hat{S}(\omega) \hat{\varphi}^*(\omega). \tag{23}$$

Since $\mathrm{supp} m_0(2\cdot) = \frac{1}{2} \mathrm{supp} m_0(\cdot)$, (23) implies

$$\prod_{j=0}^{J-1} m_0(2^j\omega) \hat{\varphi}(\omega) \chi_{\cap_{j=0}^{J-1} \mathrm{supp} m_0(2^j \cdot)}(\omega)$$

$$= \prod_{j=0}^{J-1} m_0(2^j\omega) \hat{\varphi}^*(\omega) \hat{S}(\omega), \tag{24}$$

i.e.,

$$\prod_{j=0}^{J-1} m_0(2^j\omega) \hat{\varphi}(\omega) = \prod_{j=0}^{J-1} m_0(2^j\omega) \hat{\varphi}^*(\omega) \hat{S}(\omega). \tag{25}$$

By the way,

$$\hat{\varphi}_J^*(2^J\omega) = \sum_k \hat{\varphi}(2^J\omega + 2^{J+1}k\pi) \tag{26}$$

$$= \prod_{j=0}^{J-1} m_0(2^j\omega) \sum_k \hat{\varphi}(\omega + 2k\pi) \tag{27}$$

$$= \prod_{j=0}^{J-1} m_0(2^j\omega) \hat{\varphi}^*(\omega) \tag{28}$$

and

$$\prod_{j=0}^{J-1} m_0(2^j\omega) \hat{\varphi}(\omega) = \hat{\varphi}(2^J\omega). \tag{29}$$

Take inverse Fourier transform on both sides of (22) and refer to (17), we have

$$S(t) = \sum_k c_k \varphi(t - k). \tag{30}$$

It implies $S(t) \in V_0$ due to $\{\varphi(t-k)\}_k$ is a Riesz basis of V_0. On the other hand, (25), (28) and (29) implies that

$$\hat{\varphi}(2^J\omega) = \hat{\varphi}_J^*(2^J\omega) S(\omega), \tag{31}$$

i.e.,

$$\hat{\varphi}(\omega) = \hat{\varphi}_J^*(\omega) S(2^{-J}\omega). \tag{32}$$

From Poisson summation formula,

$$\hat{\varphi}_J^*(\omega) = \sum_n \hat{\varphi}(\omega + 2^{J+1}n\pi) \tag{33}$$

$$= 2^{-J} \sum_n \varphi(2^{-J}n) e^{-in\omega/2^J}. \tag{34}$$

Take inverse Fourier Transform on both sides of (32) and refer to (34),

$$\varphi(t) = \sum_n \varphi(2^{-J}n) S(2^J t - n). \tag{35}$$

For any $f(t) \in V_0$, it can be written to be $f(t) = \sum_k a_k \varphi(t-k)$ for some $\{a_k\}_k \in l^2$. Due to (35) we have

$$f(t) = \sum_k a_k \sum_n \varphi(2^{-J}n) S(2^J(t-k) - n) \tag{36}$$

$$= \sum_k a_k \sum_l \varphi(2^{-J}l - k) S(2^J t - l) \tag{37}$$

$$= \sum_l S(2^J t - l) \sum_k a_k \varphi(2^{-J}l - k) \tag{38}$$

$$= \sum_l f(2^{-J}l) S(2^J t - l), \tag{39}$$

where (37) is due to the index transform $l = 2^J k + n$.

Step 2: necessity.

On the contrary, if there is an $S(t) \in V_0$ such that (15) holds in $L^2(R)$-sense, then

$$\varphi(t) = \sum_n \varphi(2^{-J}n) S(2^J t - n) \tag{40}$$

holds in $L^2(R)$-sense. By taking Fourier transform on both sides of (40) and referring to (34), we obtain

$$\hat{\varphi}(\omega) = \hat{\varphi}_J^*(\omega)\hat{S}(2^{-J}\omega), \tag{41}$$

$$\text{i.e., } \hat{\varphi}(2^J\omega) = \hat{\varphi}_J^*(2^J\omega)\hat{S}(\omega) \text{ or} \tag{42}$$

$$\prod_{j=0}^{J-1} m_0(2^j\omega)\hat{\varphi}(\omega) = \prod_{j=0}^{J-1} m_0(2^j\omega)\hat{\varphi}^*(\omega)\hat{S}(\omega). \tag{43}$$

(43) implies that

$$\cap_{j=0}^{J-1}\mathrm{supp}\, m_0(2^j\cdot) \cap \mathrm{supp}\,\hat{\varphi}(\cdot) \subset \cap_{j=0}^{J-1}\mathrm{supp}\, m_0(2^j\cdot) \cap \mathrm{supp}\,\hat{\varphi}^*(\cdot) \tag{44}$$

holds except for a zero measure subset of R, i.e.,

$$\cap_{j=0}^{J-1}\mathrm{supp}\, m_0(2^j\cdot) \cap \mathrm{supp}\,\hat{\varphi}(\cdot + 2k\pi) \subset \cap_{j=0}^{J-1}\mathrm{supp}\, m_0(2^j\cdot) \cap \mathrm{supp}\,\hat{\varphi}^*(\cdot) \tag{45}$$

holds for all $k \in Z$ since $m_0(\omega)$ and $\hat{\varphi}^*(\omega)$ are 2π-periodic. Meanwhile

$$\cup_k \mathrm{supp}\,\hat{\varphi}(\cdot + 2k\pi) = R \tag{46}$$

holds except for a zero measure subset of R. Otherwise there must be some nonzero measure subset $\delta \subset R$ such that $\hat{\varphi}(\omega + 2k\pi) = 0$ holds for any $\omega \in \delta$ and any $k \in Z$. Then $G_\varphi(\omega) = 0$ holds for any $\omega \in \delta$. It is contradictory to (10). Therefore (45) and (46) imply that

$$\cap_{j=0}^{J-1}\mathrm{supp}\, m_0(2^j\cdot) \cap \mathrm{supp}\,\hat{\varphi}^*(\cdot) \supset \cap_{j=0}^{J-1}\mathrm{supp}\, m_0(2^j\cdot) \tag{47}$$

holds except for a zero measure subset of R, i.e., $\mathrm{supp}\hat{\varphi}^*(\cdot) \supset \cap_{j=0}^{J-1}\mathrm{supp}\, m_0(2^j\cdot)$. Now (43) can be rewritten to be

$$\frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)}\chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega) = \hat{S}(\omega)\chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega). \tag{48}$$

Since $\hat{S}(\omega) \in L^2(R)$ due to $S(\omega) \in L^2(R)$, we derive

$$\infty > \int_R \left|\frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)}\right|^2 \chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega)d\omega \tag{49}$$

$$= \int_{-\pi}^{\pi} \frac{\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2}{|\hat{\varphi}^*(\omega)|^2} \cdot \chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega)d\omega \tag{50}$$

$$\geq \|G_\varphi^2(\omega)\|_0 \int_{-\pi}^{\pi} \left|\frac{1}{\hat{\varphi}^*(\omega)} \cdot \chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega)\right|^2 d\omega. \tag{51}$$

From (51) and (10), we can now conclude that $\frac{1}{\hat{\varphi}^*(\omega)}\chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega) \in L^2[-\pi, \pi]$ holds.

Remark

1. The so called oversampling property (see Introduction or Walter [14] or Xia [16]) is in fact the special case $\hat{S}(\omega) = \hat{\varphi}(\omega)$ for *a.e.* $\omega \in \cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)$.
2. If $\mathrm{supp}\, m_0(\cdot) \subset 2\mathrm{supp}\, m_0(\cdot)$ holds, we find $\chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega) = \chi_{2^{1-J}\mathrm{supp}\, m_0(\cdot)}(\omega)$. In many practical case, it does be the true story. But the condition $\frac{1}{\hat{\varphi}^*(\omega)}\chi_{2^{1-J}\mathrm{supp}\, m_0(\cdot)}(\omega) \in L^2(R)$ is easier to be verified.
3. The theorem can be easily modified to V_m since $f(2^{-m}t) \in V_0$ when $f(t) \in V_m$. By using formula (15), $f(2^{-m}t)$ can be recovered by

$$f(2^{-m}t) = \sum_n f(2^{-J-m}n)S(2^Jt - n), \tag{52}$$

i.e.,

$$f(t) = \sum_n f(2^{-J-m}n)S(2^{J+m}t - n) \tag{53}$$

for $f(t) \in V_m$. Since $V_m \to L^2(R)$ in $L^2(R)$-sense (refer to Boor-Devore-Ron [3]), we can approximately recover any finite energy signal $f(t) \in L^2(R)$ if the sampling is fine enough (m is big enough for fixed J).

3. Truncation Error and Aliasing Error

When oversampling theorem is applied to reconstruct signals we should know how many items we need to calculate so that the recovered signal is as close to the original one as we expected. Then the truncation error defined by

$$T_f^e(t) = \sum_{|n|>N} f(2^{-J}n)S(2^Jt - n) \tag{54}$$

for $f(t) \in V_0$ should be estimated. But we need a little stronger constraint to be imposed on scaling function than in Theorem 1.

Theorem 2: Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}_m$ such that $\{\varphi(n)\}_n \in l^2$ and $\frac{1}{\hat{\varphi}^*(\omega)}\chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\omega)} \in L^\infty[-\pi, \pi]$ for some $J \in Z^+$. Then

$$\|T_f^e(t)\| \leq 2^{-J/2}\left(\sum_{|n|\geq N}\left|f\left(\frac{n}{2^J}\right)\right|^2\right)^{1/2} \cdot \left\|\frac{G_\varphi(\omega)}{\hat{\varphi}^*(\omega)}\chi_{\cap_{j=0}^{J-1}2^{-j}\mathrm{supp}\, m_0(\cdot)}(\omega)\right\|_\infty. \tag{55}$$

Proof From Parseval Identity, we have

$$\|T_f(t)\| = \left\|\sum_{|n|\geq N} f\left(\frac{n}{2^J}\right)S(2^Jt - n)\right\| \tag{56}$$

$$= \frac{1}{\sqrt{2\pi}} \left\| 2^{-J} \sum_{|n|\geq N} f\left(\frac{n}{2^J}\right) e^{-i2^{-J}n\omega} \hat{S}\left(\frac{\omega}{2^J}\right) \right\| \tag{57}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^{2^{J+1}\pi} \left| 2^{-J} \sum_{|n|\geq N} f\left(\frac{n}{2^J}\right) e^{-i2^{-J}n\omega} \right|^2 \cdot \frac{\sum_k |\hat{\varphi}(\frac{\omega}{2^J} + 2k\pi)|^2}{|\hat{\varphi}^*(\frac{\omega}{2^J})|^2} \cdot \chi_{\cap_{j=1}^{J} 2^j \operatorname{supp} m_0(\cdot)}(\omega) d\omega \right)^{1/2} \tag{58}$$

$$\leq 2^{-J/2} \left(\sum_{|n|\geq N} \left| f\left(\frac{n}{2^J}\right) \right|^2 \right)^{1/2} \cdot \left\| \frac{G_\varphi(\frac{\omega}{2^J})}{\hat{\varphi}^*(\frac{\omega}{2^J})} \chi_{\cap_{j=1}^{J} 2^j \operatorname{supp} m_0(\cdot)}(\omega) \right\|_\infty \tag{59}$$

$$= 2^{-J/2} \left(\sum_{|n|\geq N} \left| f\left(\frac{n}{2^J}\right) \right|^2 \right)^{1/2} \cdot \left\| \frac{G_\varphi(\omega)}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \operatorname{supp} m_0(\cdot)}(\omega) \right\|_\infty . \tag{60}$$

Remark For $f(t) \in V_m$, the truncation error is calculated as

$$\|T_f^e(t)\| \leq 2^{\frac{-(J+m)}{2}} \left(\sum_{|n|\geq N} \left| f\left(\frac{n}{2^{J+m}}\right) \right|^2 \right)^{\frac{1}{2}} \cdot \left\| \frac{G_\varphi(\omega)}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \operatorname{supp} m_0(\cdot)}(\omega) \right\|_\infty . \tag{61}$$

The other error that should be estimated is the aliasing error which was proposed by Brown [1] in reconstructing a non-band-limited signals by means of the band pass sampling theorems. Beaty-Higgins [2] extended it to a more general case as to estimate the error of approximating signals by the multiplication of Shannon scaling function. It was Walter [13] who established the sampling theorem for wavelet subspaces and estimated an upper bound for the aliasing error. Chen-Itoh [5] improved Walter's results and obtained a more precise upper bound. For oversampling theorem the aliasing error is defined by

$$A_f^e(t) = f(t) - \sum_n f(2^{-J}n) S(2^J t - n) \tag{62}$$

for $f(t) \in V_1$. By calculating the aliasing error we can select the scaling functions with the smaller aliasing error to recover signals.

Theorem 3: Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}_m$ such that $\{\varphi(n)\}_n \in l^2$ and $\frac{1}{\hat{\varphi}^*(\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \operatorname{supp} m_0(\cdot)}(\omega) \in L^2[-\pi, \pi]$ for some $J \in Z^+$. Then

$$\|A_f^e(t)\| \leq 2^{J/2} \left(\sum_n |b_n|^2 \right)^{1/2} \left\| m_1(\omega) G_\varphi\left(\frac{\omega}{2^{J-1}}\right) \cdot \left(\prod_{j=1}^{J-1} m_0\left(\frac{\omega}{2^j}\right) \right)^{1-\delta_{J-1}} \right\|_\infty \tag{63}$$

where $\{b_k\}_k$ is the wavelet coefficients of $f(t)$ in W_0, and δ_{J-1} is the Dirac function, i.e.,

$$\delta_{J-1} = \begin{cases} 1, & J = 1 \\ 0, & J \neq 1. \end{cases} \tag{64}$$

Proof Let $W_0 = V_1 \ominus V_0$. Due to (15), we only need to show (63) holds for any $f \in W_0$. Let $\psi(t) \in W_0$ be the wavelet of MRA $\{V_m\}_m$. Since $\{\psi(t-k)\}_k$ forms a Riesz basis of W_0, there must be a $\{b_n\}_n \in l^2$ (the wavelet coefficients of $f(t)$ in W_0) such that

$$f(t) = \sum_n b_n \psi(t - n). \tag{65}$$

Set $C_f(\omega) = \sum_n b_n e^{-in\omega}$ and take Fourier transform on both sides of (65),

$$\hat{f}(\omega) = C_f(\omega) \hat{\psi}(\omega). \tag{66}$$

From Parseval Identity, it shows

$$\|A_f^e(t)\| = \frac{1}{\sqrt{2\pi}} \left\| C_f(\omega)\hat{\psi}(\omega) - 2^{-J} \cdot \sum_n f(2^{-J}n) e^{-i2^{-J}n\omega} \hat{S}(2^{-J}\omega) \right\|. \tag{67}$$

Since $2^{-J} \sum_n f(2^{-J}n) e^{-i2^{-J}n\omega} = \sum_n \hat{f}(\omega + 2^{J+1}n\pi)$ due to Poisson summation Formula, we derive

$$\|A_f^e(t)\| = \frac{1}{\sqrt{2\pi}} \left\| C_f(\omega)\hat{\psi}(\omega) - \sum_n \hat{f}(\omega + 2^{J+1}n\pi) \cdot \frac{\hat{\varphi}(2^{-J}\omega)}{\hat{\varphi}^*(2^{-J}\omega)} \chi_{\cap_{j=0}^{J-1} 2^{-j} \operatorname{supp} m_0(2^{-J}\cdot)}(\omega) \right\| \tag{68}$$

$$= \frac{1}{\sqrt{2\pi}} \left\| C_f(\omega) m_1\left(\frac{\omega}{2}\right) \hat{\varphi}\left(\frac{\omega}{2}\right) - \sum_n \hat{\psi}(\omega + 2^{J+1}n\pi) \cdot C_f(\omega) \frac{\hat{\varphi}(2^{-J}\omega)}{\hat{\varphi}^*(2^{-J}\omega)} \chi_{\cap_{j=0}^{J-1} 2^{J-j} \operatorname{supp} m_0(\cdot)}(\omega) \right\| \tag{69}$$

Case 1: $J = 1$. Then

$\|A_f^e(t)\|$

$$= \frac{1}{\sqrt{2\pi}} \left\| C_f(\omega) m_1\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) - C_f(\omega) m_1\left(\frac{\omega}{2}\right) \cdot \hat{\varphi}^*\left(\frac{\omega}{2}\right) \frac{\hat{\varphi}(\frac{\omega}{2})}{\hat{\varphi}^*(\frac{\omega}{2})} \chi_{2\mathrm{supp}\, m_0(\cdot)}(\omega) \right\| \tag{70}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^{4\pi} \left| C_f(\omega) m_1\left(\frac{\omega}{2}\right)\right|^2 \cdot \sum_k \left|\hat{\varphi}\left(\frac{\omega}{2} + 2k\pi\right)\right|^2 \cdot |1 - \chi_{2\mathrm{supp}\, m_0(\cdot)}(\omega)| d\omega \right)^{1/2} \tag{71}$$

$$\leqq \frac{1}{\sqrt{2\pi}} \left(\int_0^{4\pi} |C_f(\omega)|^2 d\omega \right)^{1/2} \cdot \left\| m_1\left(\frac{\omega}{2}\right) G_\varphi\left(\frac{\omega}{2}\right) (1 - \chi_{\mathrm{supp}\, m_0(\frac{\cdot}{2})}(\omega)) \right\|_\infty \tag{72}$$

$$= \sqrt{2} \left(\sum_n |b_n|^2 \right)^{1/2} \cdot \left\| m_1(\omega) G_\varphi(\omega) \chi_{R \ominus \mathrm{supp}\, m_0(\cdot)}(\omega) \right\|_\infty . \tag{73}$$

Case 2: $J \geqq 2$. Then

$$\|A_f^e(t)\| = \frac{1}{\sqrt{2\pi}} \left\| C_f(\omega) m_1\left(\frac{\omega}{2}\right) \prod_{j=2}^{J} m_0\left(\frac{\omega}{2^j}\right) \hat{\varphi}\left(\frac{\omega}{2^J}\right) - C_f(\omega) m_1\left(\frac{\omega}{2}\right) \prod_{j=2}^{J} m_0\left(\frac{\omega}{2^j}\right) \hat{\varphi}^*\left(\frac{\omega}{2^J}\right) \cdot \frac{\hat{\varphi}(2^{-J}\omega)}{\hat{\varphi}^*(2^{-J}\omega)} \chi_{\cap_{j=1}^J 2^j \mathrm{supp}\, m_0(\cdot)}(\omega) \right\| \tag{74}$$

$$= \frac{1}{\sqrt{2\pi}} \left\| C_f(\omega) m_1\left(\frac{\omega}{2}\right) \prod_{j=2}^{J} m_0\left(\frac{\omega}{2^j}\right) \hat{\varphi}\left(\frac{\omega}{2^J}\right) \cdot (1 - \chi_{\cap_{j=1}^J 2^j \mathrm{supp}\, m_0(\cdot)}(\omega)) \right\| \tag{75}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^{2^{J+1}\pi} \left| C_f(\omega) m_1\left(\frac{\omega}{2}\right) \prod_{j=2}^{J} m_0\left(\frac{\omega}{2^j}\right)\right|^2 \cdot \sum_k \left|\hat{\varphi}\left(\frac{\omega}{2^J} + 2k\pi\right)\right|^2 \cdot |1 - \chi_{\cap_{j=1}^J 2^j \mathrm{supp}\, m_0(\cdot)}(\omega)| d\omega \right)^{1/2} \tag{76}$$

$$\leqq \frac{1}{\sqrt{2\pi}} \left(\int_0^{2^{J+1}\pi} |C_f(\omega)|^2 d\omega \right)^{1/2} \cdot \left\| m_1\left(\frac{\omega}{2}\right) \prod_{j=2}^{J} m_0\left(\frac{\omega}{2^j}\right) G_\varphi\left(\frac{\omega}{2^J}\right) \cdot \left(1 - \chi_{\cap_{j=1}^J 2^j \mathrm{supp}\, m_0(\cdot)}(\omega)\right) \right\|_\infty \tag{77}$$

$$= 2^{J/2} \left(\sum_n |b_n|^2 \right)^{1/2} \left\| m_1(\omega) \prod_{j=1}^{J-1} m_0\left(\frac{\omega}{2^j}\right) \cdot G_\varphi\left(\frac{\omega}{2^{J-1}}\right) \chi_{R \ominus \cap_{j=0}^{J-1} 2^j \mathrm{supp}\, m_0(\cdot)}(\omega) \right\|_\infty . \tag{78}$$

Combine Case 1 and Case 2, we derive

$$\|A_f^e(t)\| \leqq 2^{J/2} \left(\sum_n |b_n|^2 \right)^{1/2} \left\| m_1(\omega) G_\varphi\left(\frac{\omega}{2^{J-1}}\right) \cdot \left(\prod_{j=1}^{J-1} m_0\left(\frac{\omega}{2^j}\right) \right)^{1-\delta_{J-1}} \cdot \chi_{R \ominus \cap_{j=0}^{J-1} 2^j \mathrm{supp}\, m_0(\cdot)}(\omega) \right\|_\infty$$

$$\leqq 2^{\frac{J}{2}} \left(\sum_n |b_n|^2 \right)^{\frac{1}{2}} \left\| m_1(\omega) G_\varphi\left(\frac{\omega}{2^{J-1}}\right) \cdot \left(\prod_{j=1}^{J-1} m_0\left(\frac{\omega}{2^j}\right) \right)^{1-\delta_{J-1}} \right\|_\infty . \tag{79}$$

Remark

1. In orthonormal wavelet subspaces, $G_\varphi(\omega) = 1$, $|m_0(\omega)| \leqq 1$ and $|m_1(\omega)| \leqq 1$ hold for *a.e.* $\omega \in R$. Hence the aliasing error is
$$\|A_f^e(t)\| \leqq 2^{J/2} \left(\sum_n |b_n|^2 \right)^{1/2} . \tag{80}$$
2. For $f(t) \in V_{m+1}$, we find that the aliasing error satisfies
$$\|A_f^e(t)\| \leqq 2^{\frac{J-m}{2}} \left(\sum_n |b_n|^2 \right)^{\frac{1}{2}} \left\| m_1(\omega) G_\varphi(\omega) \cdot \left(\prod_{j=1}^{J-1} m_0\left(\frac{\omega}{2^j}\right) \right)^{1-\delta_{J-1}} \right\|_\infty \tag{81}$$
But here $\{b_n\}_n$ is the wavelet coefficient of $f(t)$ in W_m. It implies that the aliasing error can be as small as we like if the sampling is fine enough (m is big enough for fixed J).

4. Shift Oversampling Theorem

As done by Janssen [9] for Walter sampling theorem, Chen-Itoh [4] for Irregular sampling and Chen-Itoh [5] for regular sampling, the shift sampling version of oversampling theorem can also be obtained by using Zak-transform (see Heil-Walnut [8], Janssen [9] and Walter [14]). Let $\varphi(t)$ be a scaling function of MRA $\{V_m\}_m$ such that $\{\varphi(n+\sigma)\}_n \in l^2$ for some $\sigma \in [0,1)$. Then Zak-transform can be defined by

$$Z_\varphi(\sigma,\omega) = \sum_n \varphi(\sigma+n)e^{-in\omega}, \quad \omega \in R. \tag{82}$$

The following is the shift sampling version of oversampling theorem. Since the procedure is strongly similar to the previous section except that $Z_\varphi(\sigma,\omega)$ takes the role of $\hat{\varphi}^*(\omega)$, we only display the result without proof here.

Theorem 4: Let $\varphi(t)$ be the scaling function of MRA$\{V_m\}_m$ such that $\{\varphi(n+\sigma)\}_n \in l^2$ for some $\sigma \in [0,1)$. Then there is an $S_\sigma(t) \in V_0$ and a $J \in Z^+$ such that

$$f(t) = \sum_n f(2^{-J}(n+\sigma))S_\sigma(2^J t - n) \tag{83}$$

holds for $f(t) \in V_0$ in $L^2(R)$-sense if and only if

$$\frac{1}{Z_\varphi(\sigma,\omega)}\chi_{\cap_{j=0}^{J-1}2^{-j}\operatorname{supp} m_0(\cdot)}(\omega) \in L^2[-\pi,\pi] \tag{84}$$

holds. In this case $\hat{S}_\sigma(\omega) = \frac{\hat{\varphi}(\omega)}{Z_\varphi(\sigma,\omega)}$ holds for $a.e.\ \omega \in \cap_{j=0}^{J-1}2^{-j}\operatorname{supp} m_0(\cdot)$.

Remark The shift sampling version of truncation error and aliasing error for oversampling theorem also need to be estimated. Since they can be obtained just by using $Z_\varphi(\sigma,\omega)$ instead of $\hat{\varphi}^*(\omega)$, we will not display them here. Of course the shift sampling in V_m for oversampling theorem can also be obtained as the previous.

5. Conclusion

Based on the above discussion, we can summary an algorithm as what follows.

1. For the scaling function $\varphi(t)$ of MRA $\{V_m\}_m$, find a $\sigma \in [0,1)$ such that $\frac{1}{Z_\varphi(\sigma,\omega)} \chi_{\cap_{j=0}^{J-1}2^{-j}\operatorname{supp} m_0(\cdot)}(\omega) \in L^2[-\pi,\pi]$.
2. Calculate the truncation error to determine how many items we should at least calculate for approximating the original signals from their discrete samples as close as we expect.
3. Recover the original signals by oversampling theorem (by formula (83)).
4. Calculate the aliasing error to recover the sampled signals in the finer resolution wavelet subspaces.

6. An Example to Show the Algorithm

Obviously the scaling functions covered by sampling theorem and oversampling property, such as Haar scaling function, Shannon scaling function, Daubechies scaling function, Meyer scaling function, etc., all can be captured by our oversampling theorem. Since they are trival cases for the oversampling theorem and can be better dealed with by the old sampling theorem or oversampling property (refer to Walter [13], Xia [16] and Chen-Itoh [5], [6]) rather than by oversampling theorem, we will not calculate those here. We now calculate the scaling function $\varphi_c(t)$ defined in the Introduction as an examples to show the algorithm.

Example $\hat{\varphi}_c^*(\omega)|_{[-\pi,\pi]} = v(\omega)\chi_{[-\frac{2\pi}{3},\frac{2\pi}{3}]}$ implies that $\hat{\varphi}_c^*(\omega) \neq 0$ does not hold. Therefore we can not apply sampling theorem to $\varphi_c(t)$ (refer to Walter [13] and Chen-Itoh [5]). Since $\hat{\varphi}_{cJ}^*(\omega) = \hat{\varphi}_c(\omega)$ and $\hat{\varphi}_c(2^{-J}\omega) = v(\omega)$ on $\operatorname{supp}\hat{\varphi}_c(\cdot)$ hold for any $J \in Z^+$, the equation $\hat{\varphi}_c(\omega) = \hat{\varphi}_{cJ}^*(\omega)\hat{\varphi}_c(2^{-J}\omega)$, i.e., the equation $\hat{\varphi}_c(\omega) = \hat{\varphi}_c(\omega)v(\omega)$ can not hold excerpt for $v(\omega) \equiv 1$. Hence we can not yet apply oversampling property to $\varphi_c(t)$ (refer to the Introduction or Xia ([16]) and Chen-Itoh [6]). But we have known $\frac{1}{\hat{\varphi}_c{}^*(\omega)}\chi_{\operatorname{supp} m_0(\cdot)}(\omega) = \frac{1}{v(\omega)}\chi_{[-\frac{2\pi}{3},\frac{2\pi}{3}]}(\omega) \in L^\infty[-\frac{2\pi}{3},\frac{2\pi}{3}]$. Thus the oversampling theorem can be used to recover signals with $J=1$ and $\sigma=0$, i.e., there is an $S(t) \in V_0$ such that $\hat{S}(\omega) = \frac{\hat{\varphi}_c(\omega)}{v(\omega)}$ holds for $a.e.\ \omega \in [-\frac{2\pi}{3},\frac{2\pi}{3}]$, i.e., $\hat{S}(\omega) = \chi_{[-\frac{2\pi}{3},\frac{2\pi}{3}]}(\omega)$ and

$$f(t) = \sum_n f(2^{-1}n)S(2t-n) \quad \text{for } f(t) \in V_0 \tag{85}$$

holds in $L^2(R)$-sense. In fact $S(t) = \frac{\sin(2\pi t/3)}{\pi t}$ holds in this case. Therefore (85) is

$$f(t) = \sum_n f\left(\frac{n}{2}\right)\frac{\sin(\frac{4\pi t}{3} - \frac{2\pi}{3}n)}{2\pi t - n\pi} \tag{86}$$

for $f(t) \in V_0$. The truncation error and aliasing error are respectively

$$\|T_f^e(t)\| \leqq \frac{1}{\sqrt{2}}\left(\sum_{|n|\geq N}\left|f\left(\frac{n}{2}\right)\right|^2\right)^{1/2} \tag{87}$$

for $f(t) \in V_0$, and

$$\|A_f^e(\omega)\| \leqq 2\,\|m_1(\omega)\|_\infty \left(\sum_n |b_n|^2\right)^{1/2} \tag{88}$$

for $f(t) \in V_1$, where $m_1(\omega)$ depends on the wavelet constructed. For orthonormal wavelet, $|m_1(\omega)| \leqq 1$ holds for $a.e.\ \omega \in R$. For sampling in V_m the truncation error and aliasing error are respectively

$$\|T_f^e(t)\| \leqq 2^{\frac{-1-m}{2}}\left(\sum_{|n|\geq N}\left|f\left(\frac{n}{2^{m+1}}\right)\right|^2\right)^{1/2} \tag{89}$$

for $f(t) \in V_m$, and

$$\|A_f^e(\omega)\| \leq 2^{\frac{2-m}{2}} \|m_1(\omega)\|_\infty \left(\sum_n |b_n|^2\right)^{1/2} \tag{90}$$

for $f(t) \in V_{m+1}$. But here $\{b_n\}_n$ are the wavelet coefficients of $f(t)$ in W_m, and $m_1(\omega)$ also depends on the wavelets constructed.

It is worth to indicating that this example can also be indirectly dealed with by the classical Shannon sampling theorem although it, as a special case of sampling theorem, can not be directly applied to the wavelet subspaces. Since the scaling function $\varphi_c(t)$ is $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$-band, it can be expressed as

$$\varphi_c(t) = \sum_n \varphi_c\left(\frac{3}{4}n\right) \frac{\sin(\frac{4\pi}{3}t - n\pi)}{\frac{4\pi}{3}t - n\pi}. \tag{91}$$

Suppose $f(t) = \sum_k a_k \varphi_c(t-k)$ for $f(t) \in V_0$. Then

$$f(t) = \sum_{k,n} a_k \varphi_c\left(\frac{3}{4}n\right) \frac{\sin(\frac{4\pi}{3}(t-k) - n\pi)}{\frac{4\pi}{3}(t-k) - n\pi} \tag{92}$$

$$= \sum_{k,n} a_k \varphi_c\left(\frac{3}{4}n\right) \frac{\sin(\frac{4\pi}{3}t - (\frac{4}{3}k + n)\pi)}{\frac{4\pi}{3}t - (\frac{4}{3}k + n)\pi} \tag{93}$$

$$= \sum_{l \in A} \sum_k a_k \varphi_c\left(\frac{l}{4} - k\right) \frac{\sin(\frac{4\pi}{3}t - \frac{l}{3}\pi)}{\frac{4\pi}{3}t - \frac{l}{3}\pi} \tag{94}$$

$$= \sum_{l \in A} f\left(\frac{l}{4}\right) \frac{\sin(\frac{4\pi}{3}t - \frac{l}{3}\pi)}{\frac{4\pi}{3}t - \frac{l}{3}\pi}, \tag{95}$$

where (94) is due to the index transform $l = 4k + 3n$, and $A = 4Z + 3Z$. From (95) we learn that the sampling step is $\frac{1}{4}$ with computing complexity $4N$ or step $\frac{3}{4}$ with complexity $\frac{4}{3}N^2$. But (86) implies that we can recover the signal by step $\frac{1}{2}$ with complexity $2N$. It is a typical difference of oversampling theorem and the previous contributions, and also the advantage of oversampling theorem versus other ways for dealing with the scaling functions such as $\varphi_c(t)$. It is also worth to indicating that Shannon sampling theorem is available only because $\varphi_c(t)$ is fortunately bandlimited. In general case it does not work while sampling theorem is not available.

References

[1] J.L.J. Brown, "On the error in reconstructing a non band-limited function by means of the bandpass sampling theorem," J. Math. Analy. Appl., vol.18, pp.75–84, 1967.

[2] M.G. Beaty and J.R. Higgins, "Aliasing and Poisson summation in the sampling theory of Paley-Wiener Space," The Journal of Fourier Analysis and Applications, vol.1, no.1, pp.67–85, 1994.

[3] D. Boor, C. Devore, and R. Ron, "The structure of finitely generated shift invariant spaces in $L^2(R^d)$," J. Functional Analysis, vol.119, pp.37–78, 1994.

[4] W. Chen and S. Itoh, "On irregular sampling in wavelet subspaces," IEICE Trans., vol.E80-A, no.7, pp.1299–1307, 1997.

[5] W. Chen and S. Itoh, "A sampling theorem for shift invariant subspace," IEEE Trans. Signal Processing, 1998.

[6] W. Chen and S. Itoh, "Signal reconstruction by scaling functions with oversampling property," Proc. IEICE-SITA, vol.20, no.2, pp.745–748, 1997.

[7] C.K. Chui, "An introduction to wavelets," Wavelet Analysis and Application, vol.1, Academic Press, New York, 1992.

[8] C.E. Heil and D.F. Walnut, "Continuous and discrete wavelet transforms," SIAM Review, vol.31, pp.628–666, 1989.

[9] M. Janssen, "The Zak-transform and sampling theorem for wavelet subspaces," IEEE Trans. Signal Processing, vol.41, no.12, pp.3360–3364, 1993.

[10] R. Long and W. Chen, "Wavelet frame packets and wavelet basis packets," The Journal of Fourier Analysis and Application, vol.3, no.3, pp.229–256, 1997.

[11] R. Long, W. Chen, and S. Yuan, "Wavelet generated by vector Multi-resolution analysis," Applied and Computational Harmonic Analysis, vol.4, pp.317–350, 1997.

[12] Y. Meyer, "Ondelette ét opérateurs I," Hermann, Paris, 1990.

[13] G.G. Walter, "A sampling theorem for wavelet subspace," IEEE Trans. Inf. Theory, vol.38, no.2, pp.881–884, 1992.

[14] G.G. Walter, "Wavelet subspaces with an oversampling property," Indaga. Math., vol.4, no.4, pp.499–507, 1993.

[15] G.G. Walter, "Wavelets and other orthogonal systems with applications," CRC Press Inc., Boca Raton, 1994.

[16] X.G. Xia, "On orthogonal wavelets with the oversampling property," The Journal of Fourier Analysis and Applications, vol.2, pp.193–199, 1994.

[17] X.G. Xia and Z. Zhang, "On sampling theorem, wavelets, and wavelet transforms," IEEE Trans. Signal Processing, vol.41, no.12, pp.3524–3534, 1993.

Wen Chen was born in Anhui, China in 1967. He received his B.S. and M.S. on Mathematical Sciences from Wuhan University, Wuhan, China in 1990 and 1993 respectively. He joined the Institute of Mathematics, Academia Sinica, Beijing, China in 1993, and visited Université de Cergy Pontoise, Paris, France, Imperial College, London, England, and University of Dundee, Dundee, Scotland in 1997. He is now studying for his D.E. at the Graduate School of Information System, University of Electro-Communications, Tokyo, Japan. His research interests cover information theory, wavelet theory and its application.

Shuichi Itoh was born in Aichi, Japan in 1942. He received his B.E., M.E., and D.E. on electrical engineering from the University of Tokyo in 1964, 1966 and 1969 respectively. He is now a professor at the Graduate School of Information System, University of Electro-Communications, Tokyo, Japan. His research interests are data compression, pattern classification and information theory. Dr. Itoh is a member of IEEE, IPSJ and the Society of Information Theory and its Application, Japan.