PAPER

Supremum of Perturbation for Irregular Sampling in Shift Invariant Subspace

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SUMMARY In the more general framework "shift invariant subspace," the paper obtains a different estimate of sampling in function subspace to our former work, by using the Frame Theory. The derived formula is easy to be calculated, and the estimate is relaxed in some shift invariant subspaces. The former work is now, however, a special case of the present.

key words: *sampling, shift invariant subspace, generating function, Zak-transform*

1. Introduction and Preliminaries

In digital signal and image processing, digital communications, etc., a continuous signals is usually represented and processed by using its discrete samples. Then a fundamental question is how to represent a signal in terms of a discrete sequence. The famous classical Shannon Sampling Theorem describes that a finite energy band-limited signal is completely characterized by their samples values. Realizing that the Shannon function $\mathrm{sinc}(t) = \sin(t)/t$ is in fact a scaling function of an MRA (Multi-Resolution Analysis)(see Appendix), Walter [18] found a sampling theorem for a class of wavelet subspaces. Following Walter [18]'s work, Janssen [13] studied the shift-sampling in Wavelet subspaces by using Zak-transform. Xia-Zhang [22] discussed the so-called sampling property. Walter [19], Xia [21] and Chen-Itoh [7], [8] studied the the more general case "oversampling." On the other hand Aldroubi-Unser [1]–[3] and Unser-Aldroubi [17] studied the sampling procedure in shift-invariant subspaces. Chen-Itoh [9] improved the works by Walter [18] and Aldroubi-Unser [3], and we found a general sampling theorem for shift-invariant subspace.

However, in many real applications samplings are not always made regularly. Sometimes the sampling steps need to be fluctuated according to the signals so as to reduce the number of samples and the computational complexity. There are also many cases where undesirable jitter exists in sampling instants. Some communication systems may suffer from the random delay due to the channel traffic congestion and encoding delay. In such cases, the system will be made to be more efficient if the irregular factor is considered. How should these irregularly sampled signals be dealt with? For the finite energy band-limited signals, a generalization of Shannon Sampling Theorem, known as the Paley-Wiener 1/4-Theorem (see Young [23]), can be used. Following the works on sampling in wavelet subspace, Liu-Walter [15], Liu [14], and Chen-Itoh-Shiki [4] extended Paley-Wiener 1/4-Theorem to a class of wavelet subspaces. But their estimate are strictive. Then Chen-Itoh-Shiki [6] introduced a function class $L^\lambda_\sigma[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$ and $0 \in [a,b] \subset [-1,1]$) and a norm $\|\cdot\|_{L^\lambda_\sigma[a,b]}$ of $L^\lambda_\sigma[a,b]$. Finally we found an irregular sampling theorem for wavelet subspaces with an $L^\lambda_\sigma[a,b]$-scaling function as the following. The definitions of $Z_\varphi(\sigma,\omega)$ and $q_\varphi(s,\sigma)\|_{L^\lambda_\sigma[a,b]}$ follow from (72) and (13) respectively.

Theorem 1: (see Chen-Itoh-Shiki [6]) Suppose a continuous $L^\lambda_\sigma[a,b]$-scaling function $\varphi(t)$ of an MRA $\{V_m\}_m$ is such that the Zak-transform $Z_\varphi(\sigma,\omega) \neq 0$ and $\varphi(t) = O(|t|^{-s})$ for $s > 1$. Then there is a $\delta_{\sigma,\varphi} \in (0,1]$ such that for any sequence $\{\delta_k\}_k \subset [-\delta_{\sigma,\varphi}, \delta_{\sigma,\varphi}] \cap [a,b]$, there is an sequence $\{S_{\sigma,k}(t)\}_k \subset V_0$ such that

$$f(t) = \sum_k f(k+\sigma+\delta_k) S_{\sigma,k}(t) \tag{1}$$

holds for any $f(t) \in V_0$ if

$$\delta_{\sigma,\varphi} < \left(\frac{\|Z_\varphi(\sigma,\omega)G_\varphi(\omega)\|_0 \|\frac{Z_\varphi(\sigma,\omega)}{G_\varphi(\omega)}\|_0}{\|q_\varphi(s,\sigma)\|_{L^\lambda_\sigma[a,b]}} \right)^{1/\lambda}. \tag{2}$$

Applying the theorem to calculating the B-spline of degree 1 scaling function $N_1(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)}$, we find $\delta_{0,N_1} < 1/2\sqrt{3}$ when $\delta_k \geq 0$ or $\delta_k \leq 0$. But Liu-Walter [15] found that the δ_{0,N_1} of the B-spline of degree 1 is $\delta_{0,N_1} < 1/2$, and they also showed that $1/2$ is the optimal upper bound of perturbation for sampling. This implies that Chen-Itoh-Shiki [6]'s result is not at least optimal.

Our purpose in this paper is trying to find the optimal $\delta_{\sigma,\varphi}$ such that (1) holds. We would like to consider the sampling in the more general framework "shift invariant subspaces." In this framework we obtain a different estimate of $\delta_{\sigma,\varphi}$ by using the Frame Theory. By applying the new result to calculating the B-spline of

Manuscript received August 17, 2000.
Manuscript revised December 11, 2000.

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degree 1, we find $\delta_{0,N_1} < 1/2$ when $\delta_k \geq 0$ or $\delta_k \leq 0$. As we mentioned, 1/2 is optimal for $N_1(t)$. Unfortunately, even by now, we can not yet show if our formula is optimal for a general generating function.

Let us now roughly introduce the shift invariant subspaces and the Frame Theory respectively. For a function $\varphi(t) \in L^2(R)$, let

$$V(\varphi) = \{\sum_k c_k\varphi(t-k) : \ \{c_k\}_k \in l^2\}. \tag{3}$$

In general $\{\varphi(t-k)\}_k$ is not a Riesz basis of $V(\varphi)$. In fact $\{\varphi(t-k)\}_k$ is a Riesz basis of $V(\varphi)$ if and only if $0 < \|G_\varphi(\omega)\|_0 \leq \|G_\varphi(\omega)\|_\infty < \infty$, where $G_\varphi(\omega) = (\sum_k |\hat{\varphi}(\omega+2k\pi)|^2)^{1/2}$, and $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(t)$ defined by $\hat{\varphi}(\omega) = \int_R \varphi(t)e^{-i\omega t}dt$. If $\{\varphi(t-k)\}_k$ is a Riesz basis of $V(\varphi)$, $\varphi(t)$ is called a generating function. The $\{\varphi(t-k)\}_k$ is an orthonormal basis of $V(\varphi)$ if and only if $G_\varphi(\omega) = 1$ (a.e.). In this case, $\varphi(t)$ is called an orthonormal generating function (see deBoor-deVore-Ron [12] and Aldroubi-Unser [3]).

A function sequence $\{S_n(t)\}_n$ is called a frame of a subspace H of $L^2(R)$ if there is a constant $C \geq 1$ such that

$$C^{-1}\|f\|^2 \leq \sum_n |\langle f(t), S_n(t)\rangle|^2 \leq C\|f\|^2 \tag{4}$$

holds for any $f(t) \in H$. Obviously a basis is a frame. Moreover there exists an unique frame $\tilde{S}_n(t)$ of H (called the dual frame of $\{S_n(t)\}_n$) such that

$$f(t) = \sum_n \langle f(t), \tilde{S}_n(t)\rangle S_n(t) \tag{5}$$

$$= \sum_n \langle f(t), S_n(t)\rangle \tilde{S}_n(t). \tag{6}$$

always holds for any $f(t) \in H$. if $\{S_n(t)\}_n$ is independent, $\{S_n(t)\}_n$ is biorthogonal to $\{\tilde{S}_n(t)\}_n$, i.e.,

$$\langle S_m(t), \tilde{S}_n(t)\rangle = \delta_{m,n}, \tag{7}$$

where $\delta_{m,n}$ is Dirac function† (see Young [23]).

The following are some notations used in this paper. For a measurable set E, $|E|$ denotes the measure of E. For the measurable functions $f(t)$ and $g(t)$, we write

$$\langle f(t), g(t)\rangle = \int_R f(t)g(t)dt, \tag{8}$$

$$\|f\| = \sqrt{\langle f(t), f(t)\rangle}, \tag{9}$$

$$\|f\|_\pi = (\int_0^{2\pi} |f(t)|^2 dt)^{1/2}, \tag{10}$$

$$\|f\|_\infty = \inf_{|E|=0} \sup_{R\backslash E} |f(t)|, \tag{11}$$

$$\|f\|_0 = \sup_{|E|=0} \inf_{R\backslash E} |f(t)| \tag{12}$$

$$q_f(s,t) = \sum_n f(s-n)f(t-n) \tag{13}$$

$$\hat{f}^*(\omega) = \sum_k f(k)e^{-ik\omega}. \tag{14}$$

2. A Sampling Theorem for Shift Invariant Subspaces

When we want to find a method to reconstruct a signal $f(t)$ by using their samples $\{f(t_k)\}_k$, obviously the samples can not be arbitrary, i.e., some constraints should be imposed on $\{f(t_k)\}_k$. The weaker the constraints is the better the reconstruction method is. Our purpose in this section is to find this kind of weak constraints. Fortunately we found a nearly necessary condition such that a reconstruction formula like (1) holds. The result will be also applied to the following sections.

Theorem 2: Suppose a generating function $\varphi(t)$ of a shift invariant subspace $V(\varphi)$ is such that $\{\varphi(t_n - k)\}_k \in l^2$ for any $n \in Z$. Then there is a frame $\{S_n(t)\}_n$ of $V(\varphi)$ such that

$$f(t) = \sum_n f(t_n)S_n(t) \tag{15}$$

holds for any $f(t) \in V(\varphi)$ if there is a constant $C \geq 1$ such that

$$C^{-1}\|f\|^2 \leq \sum_n |f(t_n)|^2 \leq C\|f\|^2 \tag{16}$$

holds for any $f(t) \in V(\varphi)$.

Proof: Take $g(t)$ such that $\hat{g}(\omega) = \hat{\varphi}(\omega)G_\varphi^{-1}(\omega)$. Then $g(t)$ is an orthonormal generating function of the shift invariant subspace $V(\varphi)$ (see deBoor-deVore-Ron [12]). Suppose $G_\varphi^{-1}(\omega) = \sum_k g_k e^{-ik\omega}$. Then $g(t) = \sum_k g_k\varphi(t-k)$, and

$$(\sum_k |g(t_n-k)|^2)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}}\|\sum_k g(t_n-k)e^{ik\omega}\|_\pi \tag{17}$$

$$= \frac{1}{\sqrt{2\pi}}\|\sum_k\sum_l g_l\varphi(t_n-k-l)e^{ik\omega}\|_\pi \tag{18}$$

$$= \frac{1}{\sqrt{2\pi}}\|\sum_l g_l e^{-il\omega}\sum_k \varphi(t_n-k-l)e^{i(k+l)\omega}\|_\pi$$

$$= \frac{1}{\sqrt{2\pi}}\|G_\varphi^{-1}(\omega)\sum_k \varphi(t_n-k)e^{ik\omega}\|_\pi \tag{19}$$

$$^\dagger\delta_{m,n} = \begin{cases} 1, & m=n \\ 0, & m \neq n \end{cases}$$

$$\leq \frac{1}{\sqrt{2\pi}}\|G_\varphi^{-1}(\omega)\|_\infty\|\sum_k \varphi(t_n-k)e^{ik\omega}\|_\pi \tag{20}$$

$$= \|G_\varphi^{-1}(\omega)\|_\infty\|(\sum_k |\varphi(t_n-k)|^2)^{1/2}. \tag{21}$$

Therefore $\{g(t_n-k)\}_k \in l^2$ due to $\{\varphi(t_n-k)\}_k \in l^2$. Let

$$q_g(t,t_n) = \sum_k g(t-k)g(t_n-k). \tag{22}$$

Then $q_g(t,t_n)$ is well defined and $q_g(t,t_n) \in V(\varphi)$ since $\{g(t-n)\}_n$ is a Riesz basis of $V(\varphi)$. For any $f(t) \in V(\varphi)$, there is a $\{c_k\}_k \in l^2$ such that $f(t) = \sum_k c_k g(t-k)$. Following the Parseval Identity, we derive

$$\langle f(t), q_g(t,t_n)\rangle \tag{23}$$

$$= \frac{1}{2\pi}\int_R \hat{f}(\omega)\overline{\hat{q}_g(\omega,t_n)}d\omega \tag{24}$$

$$= \frac{1}{2\pi}\int_R \hat{g}(\omega)\sum_k c_k e^{-ik\omega}\sum_k \overline{\hat{g}(\omega)g(t_n-k)e^{-ik\omega}}d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi} G_g^2(\omega)\sum_k c_k e^{-ik\omega}\overline{\sum_k g(t_n-k)e^{-ik\omega}}d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\sum_k c_k e^{-ik\omega}\overline{\sum_n g(t_n-k)e^{-ik\omega}}d\omega \tag{25}$$

$$= \sum_k c_k g(t_n-k) \tag{26}$$

$$= f(t_n), \tag{27}$$

where (25) is due to the fact $G_g(\omega) = 1$ (a.e.). Hence

$$C^{-1}\|f\|^2 \leq \sum_n |\langle f(t), q_g(t,t_n)\rangle|^2 \leq C\|f\|^2 \tag{28}$$

holds for any $f(t) \in V(\varphi)$. It means that $\{q_g(t,t_n)\}_n$ is a frame of $V(\varphi)$. Thus there is a dual frame $\{S_n(t)\}_n$ of $\{q_g(t,t_n)\}_n$ in $V(\varphi)$ such that

$$f(t) = \sum_n S_n(t)\langle f(t), q_g(t,t_n)\rangle \tag{29}$$

$$= \sum_n f(t_n)S_n(t) \tag{30}$$

holds for any $f(t) \in V(\varphi)$.

Remark:

1. On the contrary if $\{S_n(t)\}_n \in V(\varphi)$ is the frame such that (15) holds, there is also a dual frame $\{\tilde{S}_n(t)\}_n$ of $S_n(t)$ in $V(\varphi)$ such that

$$C^{-1}\|f\|^2 \leq \sum_n |\langle \tilde{S}_n(t), f(t)\rangle|^2 \leq C\|f\|^2 \tag{31}$$

holds for some $C \geq 1$ and any $f(t) \in V(\varphi)$. In general the dual frame is not biorthogonal to frame if the frame is not independent. Now we assume that $\{S_n(t)\}_n$ is independent. Then

$$\sum_n |\langle \tilde{S}_n(t), f(t)\rangle|^2$$

$$= \sum_n |\langle \tilde{S}_n(t), \sum_m f(t_m)S_m(t)\rangle|^2 \tag{32}$$

$$= \sum_n |\sum_m f(t_m)\langle \tilde{S}_n(t), S_m(t)\rangle|^2 \tag{33}$$

$$= \sum_n |f(t_n)|^2. \tag{34}$$

This implies that our condition (16) is also necessary if $\{S_n(t)\}_n$ is additionally independent. So we call that condition(16) is nearly necessary.

2. Another task is finding the $\{S_n(t)\}_n$ so that (15) holds. In general case, we know that $\{S_n(t)\}_n$ is a frame and biorthogonal to $\{q_g(t,t_n)\}_n$ in $V(\varphi)$. Then the $\{S_n(t)\}_n$ can be obtained by Frame Theory. If $\{S_n(t)\}_n$ is independent (this case is very often in applications), we can have the following simple result. $\{S_n(t)\}_n$ is the solution of the equations

$$\langle \hat{S}_m(\omega), \hat{\varphi}(-\omega)G_\varphi^{-2}(\omega)\sum_k \varphi(t_n-k)e^{ik\omega}\rangle$$

$$= 2\pi\delta_{n,m}. \tag{35}$$

This is because that $\{S_n(t)\}_n$ is biorthogonal to $\{q_g(t_n,t)\}_n$ and

$$\hat{q}_g(t_n,\omega)$$

$$= \sum_k g(t_n-k)\hat{g}(\omega)e^{-ik\omega} \tag{36}$$

$$= \sum_k\sum_l g_l\varphi(t_n-k-l)G_\varphi^{-1}(\omega)\hat{\varphi}(\omega)e^{-ik\omega}$$

$$= \sum_l g_l e^{il\omega}\sum_k \varphi(t_n-k)G_\varphi^{-1}(\omega)\hat{\varphi}(\omega)e^{-ik\omega}$$

$$= G_\varphi^{-2}(\omega)\hat{\varphi}(\omega)\sum_k \varphi(t_n-k)e^{-ik\omega}. \tag{37}$$

3. Irregular Sampling in Shift Invariant Subspaces

An important case of sampling is the perturbation of the regular sampling, i.e., $t_n = n + \delta_n$ ($\delta_n \in (-1,1)$). A fundamental question in this case is how to estimate the upper bound of the perturbation $\{\delta_k\}_k$. Following Paley-Wiener 1/4-Theorem for finite energy bandlimited signals, we have given an estimate for wavelet subspace by using the Riesz basis theory. In the following, we obtain a different estimate by using the Frame Theory, which is demonstrated by an example to be relaxed in some sense.

In order to establish the theorem, we also need to introduce the function class $L_\sigma^\lambda[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$, $0 \in [a,b] \subset [-1,1]$) given and used in our former work (see Chen-Itoh-Shiki [6]). We have reasoned that the class is a proper collection by giving some propositions in that paper. Here we only repeat the definition.

Definition 1: A function $f(t) \in L_\sigma^\lambda[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$ and $0 \in [a,b] \subset [-1,1]$) if there is a constant $C_{\sigma,f} > 0$ such that for any $\{\delta_k\}_k \subset [a,b]$,

$$\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)| \le C_{\sigma,f}(\sup_k |\delta_k|)^\lambda. \tag{38}$$

We also write

$$\|f\|_{L_\sigma^\lambda[a,b]} = \sup_{[a,b]} \frac{\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)|}{(\sup_k |\delta_k|)^\lambda}.$$

Theorem 3: Suppose a generating function $\varphi(t)$ of a shift invariant subspace $V(\varphi)$ is such that

1. There is a constant $C \ge 1$ such that $C^{-1} \le |\hat{\varphi}^*(\omega)| \le C$ (a.e.).
2. $\varphi(t) \in L_0^\lambda[a,b]$.

Then for any $\{\delta_k\}_k \subset [-\delta_\varphi, \delta_\varphi] \cap [a,b]$, there is a frame $\{S_k(t)\}_k$ of $V(\varphi)$ such that $f(t) = \sum_k f(k+\delta_k)S_k(t)$ holds if

$$\delta_\varphi < \left(\frac{\|\hat{\varphi}^*(\omega)\|_0}{\|\varphi\|_{L_0^\lambda[a,b]}}\right)^{1/\lambda}. \tag{39}$$

Proof: We want to apply Theorem 2 to the proof. Let $t_k = k + \delta_k$. Then we only need to show the following two items.

1. $\{\varphi(t_n - k)\}_k \in l^2$ for any $n \in Z$.
2. $C^{-1}\|f\|^2 \le \sum_k |f(t_k)|^2 \le C\|f\|^2$ holds for a constant $C \ge 1$ and for any $f(t) \in V(\varphi)$.

The condition "1" in Theorem 3 implies

$$\hat{\varphi}^*(\omega) \in L^\infty[0,2\pi] \subset L^2[0,2\pi]. \tag{40}$$

Hence

$$\sum_k |\varphi(k)|^2 = \frac{1}{2\pi}\|\hat{\varphi}^*(\omega)\|_\pi^2 < \infty. \tag{41}$$

Since

$$\begin{aligned}\left(\sum_k |\varphi(t_n-k)|^2\right)^{1/2} &\le \left(\sum_k |\varphi(n-k)|^2\right)^{1/2} \\ &\quad + \left(\sum_k |\varphi(t_n-k) - \varphi(n-k)|^2\right)^{1/2} \qquad (42)\\ &\le \left(\sum_k |\varphi(k)|^2\right)^{1/2} \\ &\quad + O\left(\left(\sum_k |\varphi(t_n-k) - \varphi(n-k)|\right)^{1/2}\right) \qquad (43)\\ &\le \left(\sum_k |\varphi(k)|^2\right)^{1/2} \\ &\quad + O\left(\|\varphi\|_{L_0^\lambda[a,b]}^{1/2} \sup_k |\delta_k|^{\lambda/2}\right), \qquad (44)\end{aligned}$$

we derive $\{\varphi(t_n-k)\}_k \in l^2$ due to $\varphi \in L_0^\lambda[a,b]$. It is exactly the "item 1." On the other hand, if we can show that there is a positive number $\theta < 1$ such that

$$\sum_k |f(t_k) - f(k)|^2 \le \theta^2 \sum_k |f(k)|^2 \tag{45}$$

holds for any $f(t) \in V(\varphi)$, then

$$(1-\theta)^2 \sum_k |f(k)|^2 \le \sum_k |f(t_k)|^2 \le (1+\theta)^2 \sum_k |f(k)|^2. \tag{46}$$

we let

$$f(t) = \sum_k c_k \varphi(t-k), \tag{47}$$

then

$$\hat{f}^*(\omega) = \hat{\varphi}^*(\omega) \sum_k c_k e^{-ik\omega}. \tag{48}$$

Together with condition "1," this means

$$C^{-1}|\sum_k c_k e^{-ik\omega}| \le |\hat{f}^*(\omega)| \le C|\sum_k c_k e^{-ik\omega}|. \tag{49}$$

Since $\{c_k\}_k$ is the coefficients of $f(t)$ expressed by basis $\varphi(t-k)$, we have

$$B^{-1}\|f\| \le \|\sum_k c_k e^{-ik\omega}\|_\pi \le B\|f\| \tag{50}$$

for some $B \ge 1$. Therefore

$$(BC)^{-1}\|f\|^2 \le \sum_k |f(k)|^2 \le BC\|f\|^2 \tag{51}$$

holds for any $f(t) \in V(\varphi)$. Facts (46) and (51) follows "item 2." In order to show (45), we let

$$\begin{aligned}\Delta &= \sum_k |f(t_k) - f(k)|^2 \qquad (52)\\ &= \sum_k |\sum_l c_l(\varphi(t_k - l) - \varphi(k-l))|^2 \qquad (53)\\ &= \sum_n \sum_{k,l} c_k c_l (\varphi(t_n-k) - \varphi(n-k)) \\ &\quad \times(\varphi(t_n-l) - \varphi(n-l)) \qquad (54)\\ &= \sum_{k,l} c_k c_l \sum_n (\varphi(t_n-k) - \varphi(n-k)) \\ &\quad \times(\varphi(t_n-l) - \varphi(n-l)). \qquad (55)\end{aligned}$$

Take $a_{k,l} = \sum_n(\varphi(t_n-k)-\varphi(n-k))(\varphi(t_n-l)-\varphi(n-l))$.

Then $a_{k,l} = a_{l,k}$ and

$$\Delta = \sum_{k,l} a_{kl} c_k c_l \tag{56}$$
$$\leq \sum_{k,l} |a_{kl}|(c_k^2 + c_l^2)/2 \tag{57}$$
$$= \sum_{k,l} |a_{kl}|(c_k^2 + c_l^2)/2 \tag{58}$$
$$= (\sum_k c_k^2 \sum_l |a_{kl}| + \sum_l c_l^2 \sum_k |a_{kl}|)/2 \tag{59}$$
$$\leq (\sum_k c_k^2) \sup_k \sum_l |a_{kl}|. \tag{60}$$

Furthermore we have

$$\sup_k \sum_l |a_{kl}|$$
$$\leq \sup_k \sum_{l,n} |(\varphi(\delta_n + n - k) - \varphi(n-k))$$
$$\times (\varphi(\delta_n + n - l) - \varphi(n-l))| \tag{61}$$
$$= \sup_k \sum_{\alpha,\beta} |(\varphi(\delta_{\alpha+k} + \alpha) - \varphi(\alpha))$$
$$\times (\varphi(\delta_{\alpha+k} + \beta) - \varphi(\beta))| \tag{62}$$
$$= \sup_k \sum_\alpha |\varphi(\delta_{\alpha+k} + \alpha) - \varphi(\alpha)|$$
$$\times \sum_\beta |\varphi(\delta_{\alpha+k} + \beta) - \varphi(\beta)| \tag{63}$$
$$\leq \|\varphi\|_{L_0^\lambda[a,b]}$$
$$\times \sup_\beta |\delta_\beta|^\lambda \sup_k \sum_\alpha |\varphi(\delta_{\alpha+k} + \alpha) - \varphi(\alpha)| \tag{64}$$
$$\leq (\|\varphi\|_{L_0^\lambda[a,b]} \sup_\beta |\delta_\beta|^\lambda)^2, \tag{65}$$

where (62) is due to the index transform $n - k = \alpha$ and $n - l = \beta$. Hence $\Delta \leq (\sum_k c_k^2)(\|\varphi\|_{L_0^\lambda[a,b]} \sup_\beta |\delta_\beta|^\lambda)^2$. Since

$$\sum_k |f(k)|^2 = \|\hat{f}^*(\omega)\|_\pi^2 \tag{66}$$
$$= \frac{1}{2\pi} \|\hat{\varphi}^*(\omega) \sum_k c_k e^{-ik\omega}\|_\pi^2 \tag{67}$$
$$\geq \frac{1}{2\pi} \|\hat{\varphi}^*\|_0^2 \| \sum c_k e^{-ik\omega}\|_\pi^2 \tag{68}$$
$$= \|\hat{\varphi}^*\|_0^2 \sum_k |c_k|^2. \tag{69}$$

Therefore we only need to show

$$\left(\sum_k c_k^2\right) (\|\varphi\|_{L_0^\lambda[a,b]} \sup_\beta |\delta_\beta|^\lambda)^2$$
$$\leq \theta \|\hat{\varphi}^*\|_0^2 \sum_k c_k^2. \tag{70}$$

It is exactly implied by (39).

Remark: The estimate in our former work is the same to (39) when $\varphi(t)$ is orthonormal. But Theorem 4 asserts that the estimate (39) holds for any generating functions. By the way, the $S_n(t)$ in the theorem is the solution of the following equations if $\{S_n(t)\}_n$ is independent.

$$\langle \hat{S}_m(\omega), \hat{\varphi}(-\omega) G_\varphi^{-2}(\omega) \sum_k \varphi(n + \delta_n - k) e^{ik\omega} \rangle$$
$$= 2\pi \delta_{m,n}. \tag{71}$$

4. Shift Sampling in Shift Invariant Subspaces

Unfortunately there are some important generating functions $\varphi(t)$'s with $\|\hat{\varphi}^*(\omega)\|_0 = 0$. An obvious example is the B-spline of degree 2, which has been calculated in our former works. As done by Janssen [13] for Walter Sampling Theorem [18], Chen-Itoh-Shiki [6] for irregular sampling theorem, we also deal with it by shift sampling. Then the shift sampling theorem can be obtained by using the Zak-transform $Z_\varphi(\sigma, \omega)$ ($\sigma \in [0, 1)$) defined by

$$Z_\varphi(\sigma, \omega) = \sum_n \varphi(\sigma + n) e^{in\omega}. \tag{72}$$

Theorem 4: Suppose a generating function $\varphi(t)$ of a shift invariant subspace $V(\varphi)$ is such that

1. There is a constant $C \geq 1$ such that $C^{-1} \leq |Z_\varphi(\sigma, \omega)| \leq C$ (a.e.).
2. $\varphi(t) \in L_\sigma^\lambda[a, b]$.

Then for any $\{\delta_k\}_k \subset [-\delta_{\sigma,\varphi}, \delta_{\sigma,\varphi}] \cap [a, b]$, there is a frame $\{S_{\sigma,k}(t)\}_k$ of $V(\varphi)$ such that (1) holds if

$$\delta_{\sigma,\varphi} < \left(\frac{\|Z_\varphi(\sigma, \omega)\|_0}{\|\varphi\|_{L_\sigma^\lambda[a,b]}} \right)^{1/\lambda}. \tag{73}$$

Remark: The $\{S_{\sigma,k}(t)\}_k$ is the solution of the following equations if $\{S_{\sigma,k}(t)\}_k$ is independent.

$$\langle \hat{S}_m(\omega), \hat{\varphi}(-\omega) G_\varphi^{-2}(\omega) \sum_k \varphi(n + \delta_n + \sigma - k) e^{ik\omega} \rangle$$
$$= 2\pi \delta_{m,n}. \tag{74}$$

5. Examples to Show the Algorithm

Since Haar function, Daubechies wavelet and Meyer wavelet are all the orthonormal generating functions (see Walter [20], Daubechies [11] and Meyer [16]), the estimate by this theorem is the same to that by our former works (refer also to the Remark of Theorem 3). We here calculate the B-spline of degree 1 and B-spline of degree 2. We find that the estimate is $\delta_{N_1} < 1/2$, which is better than our former estimate $\delta_{N_1} < 1/2\sqrt{3}$, and which is shown by Liu-Walter [15] to be the optimal. Unfortunately by now we can not yet show that the bound $\sup_{\sigma\in(-1,1)} \dfrac{\|Z_\varphi(\sigma,\omega)\|_0}{\|\varphi\|_{L^\lambda_\sigma[a,b]}}$ is optimal for all generating functions! This is an open problem.

Example 1. (see Chui [10]) B-spline of degree 1 is defined by

$$N_1(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)}. \tag{75}$$

Then $\hat{N_1}^*(\omega) = 1$. Since $\|N_1(t)\|_{L^1_0[-1,1]} = 3$, we derive $\delta_{N_1} < 1/3$. When $\delta_k \geq 0$ (or $\delta_k \leq 0$) for all $k \in Z$, the $\|N_1(t)\|_{L^1_0[-1,0]} = \|N_1(t)\|_{L^1_0[0,1]} = 2$. Therefore

$$\delta_{N_1} < 1/2. \tag{76}$$

The $\{S_n(t)\}_n$ is the solution of the equations

$$\langle \hat{S}_m(\omega), \hat{\varphi}(-\omega)G_\varphi^{-2}(\omega) \times(\delta_n e^{in\omega}+(1-\delta_n)e^{i(n-1)\omega})\rangle = 2\pi\delta_{m,n}, \tag{77}$$

where

$$G_\varphi(\omega) = (1/3 + 2/3\cos^2(\omega/2))^{1/2} \tag{78}$$

and

$$\hat{\varphi}(\omega) = (e^{-i\omega} - 1)^2/\omega^2. \tag{79}$$

The next example shows the shift sampling. It is bigger than the former estimate $1/8\sqrt{3}$, therefore, closer to the optimal estimate. But we can not show that it is optimal.

Example 2. (see Chui [10]) B-spline of degree 2 is defined by

$$N_2(t) = \frac{t^2}{2}\chi_{[0,1)} + \frac{6t-2t^2-3}{2}\chi_{[1,2)} + \frac{(3-t)^2}{2}\chi_{[2,3)}.$$

Then $\hat{N_2}^*(\omega) = e^{i\omega}(e^{i\omega}+1)/2 = 0$ when $\omega = \pi$. So we should apply the shift sampling theorem. Since $\|Z_{N_2}(\frac{1}{2},\omega)\|_0 = 1/2$, $\|N_2(t)\|_{L^1_{1/2}[-1/2,1/2]} = 3$, we derive

$$\delta_{1/2,N_2} < 1/6. \tag{80}$$

The $\{S_{1/2,n}(t)\}_n$ is the solution of the equations

$$\langle \hat{S}_{1/2,m}(\omega), \hat{N}_2(-\omega)G_{N_2}^{-2}(\omega) \times\sum_k N_2(n+\delta_n+1/2-k)e^{ik\omega}\rangle = 2\pi\delta_{m,n}. \tag{81}$$

Acknowledgment

This work is partially supported by the Japan Society for the Promotion of Sciences.

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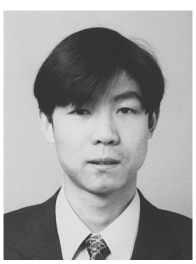
Appendix

A so-called MRA $\{V_m\}_{m\in Z}$ is a family of subspaces of $L^2(R)$, which satisfies

1. $V_m \subset V_{m+1}$, $\overline{\cup_m V_m} = L^2(R)$, and $\cap_m V_m = \{\theta\}$,
2. $f(t) \in V_m$ if and only $f(2t) \in V_{m+1}$,
3. There exists a function $\varphi(t) \in V_0$ (scaling function) such that $\{\varphi(t-n)\}_n$ forms a Riesz basis of V_0.

Each V_m is called to be a wavelet subspace. A scaling function $\varphi(t)$ is said to be orthogonal (resp. orthonormal) if $\{\varphi(x-n)\}_n$ forms an orthogonal (resp. orthonormal) basis† of V_0.

†The basis for a Hilbert space, i.e., a group of independent generating elements in a Hilbert space.



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