PAPER

The Error Estimation of Sampling in Wavelet Subspaces

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SUMMARY Following our former works on regular sampling in wavelet subspaces, the paper provides two algorithms to estimate the truncation error and aliasing error respectively when the theorem is applied to calculate concrete signals. Furthermore the shift sampling case is also discussed. Finally some important examples are calculated to show the algorithm.

key words: *sampling, scaling function, wavelet subspaces, truncation error, aliasing error*

1. Introduction and Preliminaries

Sampling is a fundamental question in signal processing, which studies how to represent a signal in terms of a discrete sequence. Shannon's popular sampling theorem states that the finite energy band-limited signals are completely characterized by their samples values. Realizing that the Shannon interpolating function $\mathrm{sinc}\,(t) = \sin(t)/t$ is in fact a scaling function of an MRA, Walter [18] found a sampling theorem for a class of wavelet subspaces.

Suppose $\varphi(t)$ is a continuous orthonormal scaling function of an MRA $\{V_m\}_{m\in Z}$ such that $|\varphi(t)| \le O((1+|t|)^{-1-\varepsilon})$ for some $\varepsilon > 0$. Let $\hat{\varphi}^*(\omega) = \sum_n \varphi(n)e^{-in\omega}$. Walter showed that there is an $S(t) \in V_0$ such that

$$f(t) = \sum_{n\in Z} f(n)S(t-n) \tag{1}$$

holds for any $f(t) \in V_0$ if $\hat{\varphi}^*(\omega) \ne 0$. Following Walter [18]'s work, Janssen [12] studied the shift sampling case by using the Zak-transform. Xia-Zhang [22] discussed the so-called sampling property ($S(t) = \varphi(t)$). Walter [19], Xia [21] and Chen-Itoh [7], [9] studied the more general case "oversampling." Liu-Walter [15], Liu [14], and Chen-Itoh-Shiki [8] even studied irregular sampling in wavelet subspaces.

On the other hand, Aldroubi-Unser [1]–[3] and Unser-Aldroubi [17] studied the sampling procedure in shift invariant subspaces. They established a more comprehensive sampling theory for shift invariant subspaces. One of their important results states that, when $\varphi(t)$ $(\in L^2(R))$ is a generating function, the orthogonal projection $g_p(t)$ of a function $g(t) \in L^2(R)$ on the shift invariant subspace $V_0(\varphi)$ is given by

$$g_p(t) = \sum_{n\in Z} \langle g(\cdot), \tilde{\varphi}(\cdot - n)\rangle \varphi(t-n), \tag{2}$$

where $\{\tilde{\varphi}(t-n)\}_n$ is the biorthogonal basis of $\{\varphi(t-n)\}_n$ in $V_0(\varphi)$, and $\langle\cdot,\cdot\rangle$ is the $L^2(R)$-inner product. They then found that the $\varphi(t)$ can be replaced by an interpolating generating function $S(t)$ if $\varphi(t) \in L^1(R) \cap L^2(R)$, $\sum_k \hat{\varphi}(\omega + 2k\pi) \ne 0$, and the Fourier transform $\hat{\varphi}(\omega)$ of $\varphi(t)$ satisfies $|\hat{\varphi}(\omega)| \le O((1+|\omega|)^{-1-\varepsilon})$ for some $\varepsilon > 0$ (see Theorem 7 in Aldroubi-Unser [3]). In fact these constraints are related to those of Walter sampling theorem due to the fact $\sum_k \hat{\varphi}(\omega+2k\pi) = \hat{\varphi}^*(-\omega)$ in some sense.

Chen-Itoh [10] improved Walter [18]'s and Aldroubi-Unser [3]'s works and found a general sampling theorem for shift invariant subspace.

Theorem 1: (**see Chen-Itoh [10]**) Suppose $\varphi(t)$ $(\in L^2(R))$ is a generating function such that the sampling $\{\varphi(n)\}_n$ makes sense and $\{\varphi(n)\}_n \in l^2$. Then there is an $S(t) \in V_0(\varphi)$ such that

$$f(t) = \sum_n f(n)S(t-n) \quad \text{for } f(t) \in V_0(\varphi) \tag{3}$$

holds in $L^2(R)$-sense if and only if

$$\frac{1}{\hat{\varphi}^*(\omega)} \in L^2[0, 2\pi] \tag{4}$$

holds. In this case $\hat{S}(\omega) = \hat{\varphi}(\omega)/\hat{\varphi}^*(\omega)$ holds for *a.e.* $\omega \in R$.

Obviously the theorem holds for wavelet subspaces. Therefore we have the following results for sampling in wavelet subspaces:

$$\begin{cases} \text{irregular sampling theorrem [8], [14], [15]} \\ \text{regular} \begin{cases} \text{sampling} \begin{cases} \text{sampling theorem [10], [18]} \\ \text{sampling property [19]} \end{cases} \\ \text{oversampling} \begin{cases} \text{oversampling theorem [7]} \\ \text{oversampling property [9].} \end{cases} \end{cases} \end{cases}$$

However, when the theorem is applied to calculate concrete signals the truncation error and aliasing error have to be estimated. In this paper we develop two algorithms to estimate them respectively.

We now drop in MRA (Multi-Resolution Analy-

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sis) and do some preparations which should be used in the following sections. A so-called MRA $\{V_m\}_{m\in Z}$ is a family of subspaces of $L^2(R)$, which satisfies

1. $V_m \subset V_{m+1}$, $\overline{\cup_m V_m} = L^2(R)$, and $\cap_m V_m = \{\theta\}$.
2. $f(t) \in V_m$ if and only $f(2t) \in V_{m+1}$.
3. There exists a function $\varphi(t) \in V_0$ (scaling function) such that $\{\varphi(t-n)\}_n$ forms a Riesz basis of V_0.

Each V_m is called to be a wavelet subspace. A scaling function $\varphi(t)$ is said to be orthogonal (resp. orthonormal) if $\{\varphi(x-n)\}_n$ forms an orthogonal (resp. orthonormal) basis of V_0.

Let $\varphi(t)$ be a scaling function of MRA $\{V_m\}_m$. Then $\{\varphi(2t-n)\}_{n\in Z}$ is a Riesz basis of V_1. Therefore there is a sequence $\{h_k\}_k \in l^2$ such that

$$\varphi(t) = \sum_k h_k \varphi(2t-k). \tag{5}$$

Let $W_0 = V_1 \ominus V_0$ be the direct complement of V_0 in V_1, and $\psi(t) \in W_0$. Then there is a sequence $\{g_k\}_k \in l^2$ such that

$$\psi(t) = \sum_k g_k \varphi(2t-k). \tag{6}$$

If $\{\psi(t-k)\}_k$ is a Riesz basis of W_0, $\psi(t)$ is said to be the wavelet of MRA$\{V_m\}_m$. Therefore for $f(t) \in V_{m+1} = W_m \oplus V_m$, there must be $\{a_n\}_n \in l^2$ and $\{b_n\}_n \in l^2$ such that

$$f(t) = \sum_n a_n \varphi(2^m t - n) + \sum_n b_n \psi(2^m t - n), \tag{7}$$

where $\{b_n\}_n$ is called the wavelet coefficients of $f(t)$ in W_m. By the way (5) and (6) also imply that

$$\hat{\varphi}(\omega) = m_0(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}) \tag{8}$$

and

$$\hat{\psi}(\omega) = m_1(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}), \tag{9}$$

where $m_0(\omega) = \frac{1}{2}\sum_k c_k e^{-ik\omega}$ and $m_1(\omega) = \frac{1}{2}\sum_k d_k e^{-ik\omega}$. Take

$$G_\varphi(\omega) = (\sum_n |\hat{\varphi}(\omega + 2n\pi)|^2)^{1/2}. \tag{10}$$

It is well known (see Meyer [16]) that

$$0 < \|G_\varphi(\omega)\|_0 \le \|G_\varphi(\omega)\|_\infty < \infty \tag{11}$$

always holds, and $\varphi(t)$ is orthonormal if and only if $G_\varphi(\omega) = 1$ holds for *a.e.* $\omega \in R$.

Finally we introduce some notations used in this paper. For measurable subset $E \subset R$, we write $|E|$ to be the measure of E. For measurable function $f(t)$, we write

$$\|f(t)\| = \left(\int_R |f(t)|^2\right)^{1/2}, \tag{12}$$

$$\|f(t)\|_0 = \sup_{|E|=0} \inf_{R\setminus E} |f(t)|, \tag{13}$$

$$\|f(t)\|_\infty = \inf_{|E|=0} \sup_{R\setminus E} |f(t)|, \tag{14}$$

$$\chi_E(t) = \begin{cases} 1 & t \in E \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

where $\chi_E(t)$ is called the characteristic function of set E.

2. Truncation Error

When sampling theorem is applied to recover the original signal $f(t)$ from their discretely sampled values $\{f(n)\}_n$ practically, we must know how many items we should at least calculate so that the recovered signal is close to the original one as we expected. Therefore we should estimate the truncation error defined by

$$T_f^e(t) = \sum_{n\ge N} f(n)S(t-n) \quad \text{for} f(t) \in V_0. \tag{16}$$

In this section, we need a little stronger constraints to be imposed on scaling function $\varphi(t)$ than in Theorem 1. But it is still very weaker than that of Walter [18] and captures many important cases such as Haar scaling function and Shannon scaling function. In fact we can show that the imposed constraints can be satisfied if Walter's conditions are given.

Theorem 2: Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}_m$ such that $\{\varphi(n)\}_n \in l^2$ and $\frac{1}{\hat{\varphi}^*(\omega)} \in L^\infty[0, 2\pi]$. Then the truncation error is bounded by

$$\|T_f^e(t)\| \le \left(\sum_{n\ge N} |f(n)|^2\right)^{1/2} \left\|\frac{G_\varphi(\omega)}{\hat{\varphi}^*(\omega)}\right\|_\infty. \tag{17}$$

This bound can also be reached in some wavelet subspaces.

Proof From Parseval identity,

$$\|T_f^e(t)\| \tag{18}$$

$$= \left\|\sum_{|n|\ge N} f(n)S(t-n)\right\| \tag{19}$$

$$= \frac{1}{\sqrt{2\pi}}\left\|\sum_{|n|\ge N} f(n)e^{-in\omega}\hat{S}(\omega)\right\| \tag{20}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_0^{2\pi}\left|\sum_{|n|\ge N} f(n)e^{-in\omega}\right|^2 \cdot \sum_k |\hat{S}(\omega+2k\pi)|^2 d\omega\right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} \left| \sum_{|n|\geq N} f(n) e^{-in\omega} \right|^2 \cdot \frac{\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2}{|\hat{\varphi}^*(\omega)|^2} d\omega \right)^{1/2} \tag{21}$$

$$\leq \left(\sum_{|n|\geq N} |f(n)|^2 \right)^{1/2} \left\| \frac{G_\varphi(\omega)}{\hat{\varphi}^*(\omega)} \right\|_\infty . \tag{22}$$

Take $\varphi(t)$ to be the orthonormal cardinal scaling functions (see Xia [21] or Walter [20]), for example $\varphi(t) = \frac{\sin \pi t}{\pi t}$. It is easy to see that (22) becomes

$$\|T_f^e(t)\| = \left(\sum_{n \geq N} |f(n)|^2 \right)^{1/2}, \tag{23}$$

i.e., the bound can be reached in the wavelet subspaces with the orthonormal cardinal scaling functions.

Remark When the sampling step is not 1 but 2^{-m}, the truncation error is defined as

$$T_f^e(t) = \sum_{n \geq N} f(n/2^m) S(2^m t - n) \tag{24}$$

for $f(t) \in V_m$. Then it can be calculated to be bounded as

$$\|T_f^e(t)\| \leq 2^{-\frac{m}{2}} \left(\sum_{n \geq N} |f(2^{-m} n)|^2 \right)^{\frac{1}{2}} \left\| \frac{G_\varphi(\omega)}{\hat{\varphi}^*(\omega)} \right\|_\infty . \tag{25}$$

It can be also reached in the wavelet subspaces with the orthonormal cardinal scaling functions.

3. Aliasing Error

The other error which should be estimated is the aliasing error which was proposed by Brown [4] in reconstructing a non-band-limited function by means of the band pass sampling theorems. Beaty-Higgin [5] extended it to a more general case as to estimate the error of approximating signals by the multiplication of Shannon scaling function. It was Walter [18] who established a sampling theorem for wavelet subspaces and estimated an upper bound for the aliasing error defined by

$$A_f^e(t) = f(t) - \sum_n f(n) S(t - n) \tag{26}$$

for $f(t) \in V_1$. But Walter [18]'s bound is not precise. It should be estimated again.

Theorem 3: Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}_m$ such that $\hat{\varphi}^*(\omega)$, $\frac{1}{\hat{\varphi}^*(\omega)} \in L^\infty[0, 2\pi]$. Then the aliasing error is bounded by

$$\|A_f^e(t)\| \leq \sqrt{2} \left(\sum_n |b_n|^2 \right)^{1/2} \cdot \left\| \frac{\hat{\varphi}^*(\omega + \pi)}{\hat{\varphi}^*(2\omega)} G_\varphi(\omega) \det M_\varphi(\omega) \right\|_\infty . \tag{27}$$

where $\{b_k\}_k$ are the wavelet coefficients of $f(t)$ in W_0,

$$M_\varphi(\omega) = \begin{pmatrix} m_0(\omega) & m_0(\omega + \pi) \\ m_1(\omega) & m_1(\omega + \pi) \end{pmatrix}, \tag{28}$$

and $\det M_\varphi(\omega)$ is the determinant of $M_\varphi(\omega)$. Furthermore the bound can also be reached in some wavelet subspaces.

Proof Step 1: To estimate the aliasing error. Since $f (\in V_1)$ can be decomposed to be $f = f_1 + f_0$, where $f_0 \in V_0$, $f_1 \in W_0 = V_1 \ominus V_0$, we only need to show (27) holds for $f_1 \in W_0$. Let $\psi(t) \in W_0$ be the wavelet of MRA $\{V_m\}_m$. Since $\{\psi(t-k)\}_k$ forms a Riesz basis of W_0, there must be a $\{b_n\}_n \in l^2$ such that

$$f_1(t) = \sum_n b_n \psi(t - n). \tag{29}$$

Set $C_{f_1}(\omega) = \sum_n b_n e^{-in\omega}$ and take Fourier transform on both sides of (29),

$$\hat{f}_1(\omega) = C_{f_1}(\omega) \hat{\psi}(\omega). \tag{30}$$

From Parseval identity, it shows

$$\|A_f^e(t)\| = \frac{1}{\sqrt{2\pi}} \left\| C_{f_1}(\omega)\hat{\psi}(\omega) - \sum_n f_1(n) e^{-in\omega} \hat{S}(\omega) \right\| \tag{31}$$

In the following (8) and (9) will be frequently used. Since $\sum\limits_n f_1(n) e^{-in\omega} = \sum\limits_n \hat{f}_1(\omega + 2n\pi)$, we derive

$$\begin{aligned}
&\|A_f^e(t)\| \\
&= \frac{1}{\sqrt{2\pi}} \left\| C_{f_1}(\omega)\hat{\psi}(\omega) - \sum_n \hat{f}_1(\omega + 2n\pi) \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)} \right\| \\
&= \frac{1}{\sqrt{2\pi}} \left\| C_{f_1}(\omega) m_1(\frac{\omega}{2}) \hat{\varphi}(\frac{\omega}{2}) - C_{f_1}(\omega) \sum_n \hat{\psi}(\omega + 2n\pi) \frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)} \right\| \qquad (32) \\
&= \frac{1}{\sqrt{2\pi}} \left\| C_{f_1}(\omega) \left(m_1(\frac{\omega}{2}) \hat{\varphi}(\frac{\omega}{2}) - \frac{\sum_n \hat{\psi}(\omega + 2n\pi)}{\hat{\varphi}^*(\omega)} m_0(\frac{\omega}{2}) \hat{\varphi}(\frac{\omega}{2}) \right) \right\| \qquad (33)
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^{4\pi} \sum_n |\hat{\varphi}(\frac{\omega}{2} + 2n\pi)|^2 |C_{f_1}(\omega)|^2 \cdot \left| m_1(\frac{\omega}{2}) - \frac{\sum_n \hat{\psi}(\omega + 2n\pi)}{\hat{\varphi}^*(\omega)} m_0(\frac{\omega}{2}) \right|^2 d\omega \right)^{1/2} \quad (34)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^{4\pi} G_\varphi^2(\frac{\omega}{2}) |C_{f_1}(\omega)|^2 \left| m_1(\frac{\omega}{2}) - \frac{\sum_n \hat{\psi}(\omega + 2n\pi)}{\hat{\varphi}^*(\omega)} m_0(\frac{\omega}{2}) \right|^2 d\omega \right)^{1/2}. \quad (35)$$

On the other hand,

$$\sum_n \hat{\psi}(\omega + 2n\pi) = \sum_n m_1(\frac{\omega}{2} + n\pi)\hat{\varphi}(\frac{\omega}{2} + n\pi) \quad (36)$$

$$= \sum_k m_1(\frac{\omega}{2} + 2k\pi)\hat{\varphi}(\frac{\omega}{2} + 2k\pi) + \sum_k m_1(\frac{\omega}{2} + (2k+1)\pi)\hat{\varphi}(\frac{\omega}{2} + (2k+1)\pi) \quad (37)$$

$$= m_1(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}) + m_1(\frac{\omega}{2} + \pi)\hat{\varphi}^*(\frac{\omega}{2} + \pi), \quad (38)$$

and

$$\hat{\varphi}^*(\omega) = \sum_n \hat{\varphi}(\omega + 2n\pi) \quad (39)$$

$$= m_0(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}) + m_0(\frac{\omega}{2} + \pi)\hat{\varphi}^*(\frac{\omega}{2} + \pi). \quad (40)$$

Then (35), (38) and (40) imply

$$\|A_f^e(t)\| = \frac{1}{\sqrt{2\pi}} \left(\int_0^{4\pi} G_\varphi^2(\frac{\omega}{2}) |C_{f_1}(\omega)|^2 \left| m_1(\frac{\omega}{2}) - \frac{m_1(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}) + m_1(\frac{\omega}{2}+\pi)\hat{\varphi}^*(\frac{\omega}{2}+\pi)}{m_0(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}) + m_0(\frac{\omega}{2}+\pi)\hat{\varphi}^*(\frac{\omega}{2}+\pi)} \cdot m_0(\frac{\omega}{2}) \right|^2 d\omega \right)^{1/2} \quad (41)$$

$$\leq \frac{1}{\sqrt{2\pi}} \left(\int_0^{4\pi} |C_{f_1}(\omega)|^2 \right)^{1/2} \left\| G_\varphi^2(\frac{\omega}{2}) \cdot \frac{m_1(\frac{\omega}{2})m_0(\frac{\omega}{2}+\pi) - m_1(\frac{\omega}{2}+\pi)m_0(\frac{\omega}{2})}{m_0(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}) + m_0(\frac{\omega}{2}+\pi)\hat{\varphi}^*(\frac{\omega}{2}+\pi)} \cdot \hat{\varphi}^*(\frac{\omega}{2} + \pi) \right\|_\infty \quad (42)$$

$$= \sqrt{2} \left(\sum_n |b_n|^2 \right)^{1/2} \left\| G_\varphi^2(\frac{\omega}{2}) \cdot \frac{m_1(\frac{\omega}{2})m_0(\frac{\omega}{2}+\pi) - m_1(\frac{\omega}{2}+\pi)m_0(\frac{\omega}{2})}{m_0(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}) + m_0(\frac{\omega}{2}+\pi)\hat{\varphi}^*(\frac{\omega}{2}+\pi)} \cdot \hat{\varphi}^*(\frac{\omega}{2} + \pi) \right\|_\infty. \quad (43)$$

By the way,

$$m_1(\frac{\omega}{2})m_0(\frac{\omega}{2}+\pi) - m_1(\frac{\omega}{2}+\pi)m_0(\frac{\omega}{2}) \quad (44)$$

$$= -\det \begin{pmatrix} m_0(\frac{\omega}{2}) & m_0(\frac{\omega}{2}+\pi) \\ m_1(\frac{\omega}{2}) & m_1(\frac{\omega}{2}+\pi) \end{pmatrix} \quad (45)$$

$$= -\det M_\varphi(\frac{\omega}{2}). \quad (46)$$

Therefore (43) becomes to be

$$\|A_f^e(t)\| \leq \sqrt{2} \left(\sum_n |b_n|^2 \right)^{1/2} \left\| \frac{G_\varphi(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}+\pi) \det M_\varphi(\frac{\omega}{2})}{m_0(\frac{\omega}{2})\hat{\varphi}^*(\frac{\omega}{2}) + m_0(\frac{\omega}{2}+\pi)\hat{\varphi}^*(\frac{\omega}{2}+\pi)} \right\|_\infty. \quad (47)$$

Together with (40), we conclude

$$\|A_f^e(t)\| \leq \sqrt{2} \left(\sum_n |b_n|^2 \right)^{1/2} \cdot \left\| \frac{\hat{\varphi}^*(\frac{\omega}{2}+\pi)}{\hat{\varphi}^*(\omega)} G_\varphi(\frac{\omega}{2}) \det M_\varphi(\frac{\omega}{2}) \right\|_\infty \quad (48)$$

$$= \sqrt{2} \left(\sum_n |b_n|^2 \right)^{1/2} \cdot \left\| \frac{\hat{\varphi}^*(\omega+\pi)}{\hat{\varphi}^*(2\omega)} G_\varphi(\omega) \det M_\varphi(\omega) \right\|_\infty. \quad (49)$$

Step 2: To show that the bound can be reached. Suppose $\varphi(t)$ be an orthonormal cardinal scaling function (see Xia [21] and Walter [20]), for example $\varphi(t) = \frac{\sin \pi x}{\pi x}$. Then $\hat{\varphi}^*(\omega) = G_\varphi(\omega) = 1$ holds for *a.e.* $\omega \in R$ and $M_\varphi(\omega)$ is unitary. By referring to the argument in Step 1, it is easy to find

$$\|A_f^e(t)\| = \sqrt{2} \left(\sum_n |b_k|^2 \right)^{1/2}. \quad (50)$$

It implies that the bound can be reached in the wavelet subspaces with the orthonormal cardinal scaling function.

Remark Compared to Walter [18]'s bound $C(\sum_n |b_n|^2)^{1/2}$ for some constant $C > 0$, we can now be sure $C = \sqrt{2}\|\frac{\hat{\varphi}^*(\omega+\pi)}{\hat{\varphi}^*(2\omega)} G_\varphi(\omega) \det M_\varphi(\omega)\|_\infty$. When $\varphi(t)$ is orthonormal, $C = \sqrt{2}\|\frac{\hat{\varphi}^*(\omega+\pi)}{\hat{\varphi}^*(2\omega)}\|_\infty$. On the other hand for sampling in V_m, the aliasing error is defined by

$$A_f^e(t) = f(t) - \sum_n f(2^{-m}n)S(2^m t - n) \tag{51}$$

for $f(t) \in V_{m+1}$. By the calculation similar to that of the sampling in V_0, we find that the bound is

$$\|A_f^e(t)\| \le 2^{\frac{1-m}{2}} \left(\sum_n |b_n|^2\right)^{1/2} \cdot \left\| \frac{\hat{\varphi}^*(\omega+\pi)}{\hat{\varphi}^*(2\omega)} G_\varphi(\omega) \det M_\varphi(\omega) \right\|_\infty . \tag{52}$$

But here $\{b_k\}_k$ is the wavelet coefficients of $f(t)$ in W_m.

4. Shift Sampling in Wavelet Subspaces

If the constraint $\frac{1}{\hat{\varphi}^*(\omega)} \in L^2[0,2\pi]$ can be satisfied, we only need to apply Theorem 1 to deal with the regularly sampled signals in wavelet subspaces. Unfortunately some scaling functions, even some important scaling functions, do not show the property. For example, take the B-spline of order 2 scaling function

$$N_2(t) = \frac{t^2}{2}\chi_{[0,1)}(t) + \frac{6t-2t^2-3}{2}\chi_{[1,2)}(t) + \frac{(3-t)^2}{2}\chi_{[2,3)}(t), \tag{53}$$

where $\chi_{[j,j+1)}(t)$ is the characteristic function of the interval $[j, j+1)$ for $j = 0,1,2$. Then $\hat{N_2}^*(\omega) = \frac{1}{2}e^{i\omega}(e^{i\omega}+1)$. Obviously $\frac{1}{\hat{N_2}^*(\omega)} = \frac{2}{e^{i\omega}(e^{i\omega}+1)}$ is not in $L^2[0,2\pi]$. Chen-Itoh [10] solves it by using Zak-transform (see Heil-Walnut [11], Janssen [12] and Walter [19]).

Suppose $\varphi(t)$ be a scaling function of MRA $\{V_m\}_m$ such that $\{\varphi(n+\sigma)\}_n \in l^2$ for some $\sigma \in [0,1)$. Then we can define Zak-transform

$$Z_\varphi(\sigma,\omega) = \sum_n \varphi(\sigma+n)e^{-in\omega}, \quad \omega \in R, \tag{54}$$

in $L^2[0,2\pi]$-sense. For the above B-spline of order 2 scaling function $N_2(t)$, we find $Z_{N_2}(\frac{1}{2},\omega) = (1+6e^{i\omega}+e^{2i\omega})/8$. Thus $\frac{1}{2} \le Z_{N_2}(\frac{1}{2},\omega)$. Obviously $\frac{1}{Z_{N_2}(\frac{1}{2},\omega)} \in L^2[0,2\pi]$ holds. Therefore we obtained the following similar result [10].

Theorem 4: Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}_m$ such that $\{\varphi(n+\sigma)\}_n \in l^2$ for some $\sigma \in [0,1)$. Then there is an $S_\sigma(t) \in V_0$ such that

$$f(t) = \sum_n f(n+\sigma)S_\sigma(t-n) \quad \text{for } f(t) \in V_0 \tag{55}$$

holds in L^2-sense if and only if

$$\frac{1}{Z_\varphi(\sigma,\omega)} \in L^2[0,2\pi] \tag{56}$$

holds. In this case $\hat{S}_\sigma(\omega) = \frac{\hat{\varphi}(\omega)}{Z_\varphi(\sigma,\omega)}$ holds for $a.e.\ \omega \in R$.

Accordingly the shift sampling versions of truncation error and aliasing error for sampling theorem can be obtained from the results obtained in the previous sections.

Theorem 5: Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}_m$ such that $Z_\varphi(\sigma,\omega)$, $\frac{1}{Z_\varphi(\sigma,\omega)} \in L^\infty[0,2\pi]$. Then the truncation error and the aliasing error are respectively bounded by

$$\|T_f^e(t)\| \le \left(\sum_{n\ge N} |f(n)|^2\right)^{1/2} \left\| \frac{G_\varphi(\omega)}{Z_\varphi(\sigma,\omega)} \right\|_\infty . \tag{57}$$

and

$$\|A_f^e(t)\| \le \sqrt{2}\left(\sum_n |b_n|^2\right)^{1/2} \left\| \frac{Z_\varphi(\sigma,\omega+\pi)}{Z_\varphi(\sigma,2\omega)} \cdot G_\varphi(\omega) \det M_\varphi(\omega) \right\|_\infty . \tag{58}$$

where $\{b_k\}_k$ are the wavelet coefficients of $f(t)$ in W_0,

$$M_\varphi(\omega) = \begin{pmatrix} m_0(\omega) & m_0(\omega+\pi) \\ m_1(\omega) & m_1(\omega+\pi) \end{pmatrix}, \tag{59}$$

and $\det M_\varphi(\omega)$ is the determinant of $M_\varphi(\omega)$. Furthermore the bound can also be reached in some wavelet subspaces.

Remark The shift sampling results for sampling in V_m can be also obtained from the results obtained in the previous sections.

5. Conclusion and Examples

Based on the above discussion, we can summarize an algorithm as what follows.

1. For the scaling function $\varphi(t)$ of MRA $\{V_m\}_m$, find a $\sigma \in [0,1)$ such that $\frac{1}{Z_\varphi(\sigma,\omega)} \in L^2[0,2\pi]$.
2. Calculate the truncation error to determine how many iterms we should calculate for approximating the original signals from their discrete samples as close as we expect.
3. Recover the original signals by sampling theorem (by Formula (55)).
4. Calculate the aliasing error to recover the sampled signals in the finer resolution wavelet subspaces.

Now we apply the algorithm to calculate some important scaling functions as examples that can be found in Chui [6], Meyer [16] and Walter [18].

Example 1 Haar scaling function $\varphi(t) = \chi_{[0,1)}$. Since $\frac{1}{\hat{\varphi}^*(\omega)} = 1 \in L^2[0,2\pi]$, we have $S(t) = \chi_{[0,1)}$ and the sampling theorem can be applied. The truncation error and aliasing error is respectively

$$\|T_f^e(t)\| = \left(\sum_{n\ge N} |f(n)|^2\right)^{1/2}, \tag{60}$$

and

$$\|A_f^e(t)\| = \sqrt{2}\left(\sum_n |b_n|^2\right)^{1/2}. \qquad (61)$$

Example 2 Shannon scaling function $\varphi(t) = \frac{\sin \pi t}{\pi t}$. Since $\frac{1}{\hat{\varphi}^*(\omega)} = 1 \in L^2[0, 2\pi]$, we have $S(t) = \frac{\sin \pi t}{\pi t}$ and the sampling theorem can be applied. The truncation error and aliasing error are the same as (60) and (61) respectively.

Example 3 B-spline of order 1 scaling function $\varphi(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)}$. Since $\frac{1}{\hat{\varphi}^*(\omega)} = e^{-i\omega} \in L^2[0, 2\pi]$, we derive $\hat{S}(\omega) = e^{-i\omega}\hat{\varphi}(\omega)$. Therefore

$$S(t) = \varphi(t-1) = (t-1)\chi_{[1,2)} + (3-t)\chi_{[2,3)}. \quad (62)$$

Since $G_\varphi(\omega) = (\frac{1}{3}(1 + 2\cos^2 \frac{\omega}{2}))^{1/2}$, we derive

$$\begin{aligned}
&\|T_f^e(t)\| \\
&\le \left(\sum_{n\ge N} |f(n)|^2\right)^{1/2} \sup_\omega \sqrt{\frac{1}{3}\left(1 + 2\cos^2\frac{\omega}{2}\right)} \\
&= \left(\sum_{n\ge N} |f(n)|^2\right)^{1/2}, \qquad (63)
\end{aligned}$$

and

$$\begin{aligned}
&\|A_f^e(t)\| \\
&\le \sqrt{2}\left(\sum_n |b_n|^2\right)^{1/2} \\
&\quad \cdot \sup_\omega \left|\sqrt{\frac{1}{3}\left(1 + 2\cos^2\frac{\omega}{2}\right)} \det M_\varphi(\omega)\right| \\
&= \sqrt{2}\left(\sum_{n\ge N} |f(n)|^2\right)^{1/2} \sup_\omega |\det M_\varphi(\omega)|, \qquad (64)
\end{aligned}$$

where $M_\varphi(\omega)$ depends on the wavelet constructed (see Jia-Shen [13]). For orthonormal wavelet, $M_\varphi(\omega)$ is unitary.

Example 4 B-spline of order 2 scaling function $N_2(t) = \frac{t^2}{2}\chi_{[0,1)}(t) + \frac{6t-2t^2-3}{2}\chi_{[1,2)}(t) + \frac{(3-t)^2}{2}\chi_{[2,3)}(t)$. Referring to Sect. 4, we have to use shift sampling. Since $Z_{N_2}(\frac{1}{2}, \omega) = (1 + 6e^{i\omega} + e^{2i\omega})/8$, we derive

$$\hat{S}_{\frac{1}{2}}(\omega) = 8\left(\frac{1 - e^{-i\omega}}{i\omega}\right)^3 / (1 + 6e^{i\omega} + e^{2i\omega}). \qquad (65)$$

By using $\|G_{N_2}(\omega)\|_\infty = 1$, the truncation error and aliasing error can be estimated as

$$\begin{aligned}
&\|T_f^e(t)\| \\
&\le \left(\sum_{n\ge N} |f(n)|^2\right)^{1/2} \sup_\omega \frac{1}{|(1 + 6e^{i\omega} + e^{2i\omega})/8|} \\
&= 2\left(\sum_{n\ge N} |f(n)|^2\right)^{1/2}. \qquad (66)
\end{aligned}$$

and

$$\|A_f^e(t)\| \le 2\sqrt{2}\left(\sum_n |b_n|^2\right)^{1/2} \sup_\omega |\det M_\varphi(\omega)|,$$

where $M_\varphi(\omega)$ also depends on the wavelet constructed

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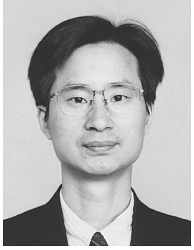

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