LETTER

An Estimate of Irregular Sampling in Wavelet Subspace

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SUMMARY The paper obtains an algorithm to estimate the irregular sampling in wavelet subspaces. Compared to our former work on the problem, the new estimate is relaxed for some wavelet subspaces.
key words: *irregular sampling, wavelet subspace, scaling function, orthogonality, biorthogonality, Zak-transform*

1. Introduction and Preliminaries

In digital signal and image processing, digital communications, etc., a continuous signal is usually represented and processed by using its discrete samples. Then a fundamental question is how to represent a signal in terms of a discrete sequence. The famous classical Shannon Sampling Theorem describes that a finite energy band-limited signal is completely characterized by their samples values. Realizing that the Shannon function $\text{sinc}(t) = \sin(t)/t$ is in fact a scaling function of an MRA, Walter[20] found a sampling theorem for a class of wavelet subspaces. Following Walter[20]'s work, Janssen[12] studied the shift-sampling in wavelet subspaces by using Zak-transform. Xia-Zhang[24] discussed the so-called sampling property. Walter[21], Xia[23] and Chen-Itoh[7],[8] studied the the more general case oversampling. On the other hand Aldroubi-Unser[1]–[3] and Unser-Aldroubi[19] studied the sampling procedure in shift-invariant subspaces. Chen-Itoh[9] improved Walter[20] and Aldroubi-Unser[3]'s works, and we found a general sampling theorem for shift-invariant subspace.

However, in many real applications samplings are not always made regularly. Sometimes the sampling steps need to be fluctuated according to the signals so as to reduce the number of samples and the computation complexity. There are also many cases where undesirable jitter exists in sampling instants. Some communication systems may suffer from the random delay due to the channel traffic congestion and encoding delay. In such cases, the system will be made to be more efficient if the irregular factor is considered. Then how are these irregularly sampled signals dealt with? For the finite energy band-limited signals, a generalization of Shannon Sampling Theorem, known as the Paley-Wiener 1/4-Theorem (see Young[25]), can be used. Following the works on sampling in wavelet subspaces, Liu-Walter[14], Liu[13], and Chen-Itoh-Shiki[4] extended Paley-Wiener 1/4-Theorem to a class of wavelet subspaces. But their results are not mild. Then Chen-Itoh-Shiki[5] introduced a function class $L^{\lambda}_{\sigma}[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$ and $0 \in [a,b] \subset [-1,1]$) and a norm $\|\cdot\|_{L^{\lambda}_{\sigma}[a,b]}$ of $L^{\lambda}_{\sigma}[a,b]$. Finally we found an irregular sampling theorem for wavelet subspaces with an $L^{\lambda}_{\sigma}[a,b]$-scaling function. Chen-Itoh[6] improved it and obtained the following result. We always use $\sum_k$ to stand for $\sum_{k=-\infty}^{\infty}$ in this paper.

Theorem 1: (see Chen-Itoh[6]) Suppose an $L^{\lambda}_{\sigma}[a,b]$-scaling function $\varphi(t)$ of an MRA $\{V_m\}_m$ is such that $C^{-1} \leqq |\sum_k \varphi(k+\sigma)e^{-ik\omega}| \leqq C$ (a.e.) for some constant $C \geqq 1$, and such that $\{\varphi(n+\sigma)\}_n \in l^2$. Then there is a $\delta_{\sigma,\varphi} \in (0,1]$ such that for any sequence $\{\delta_k\}_k \subset [-\delta_{\sigma,\varphi}, \delta_{\sigma,\varphi}]$, there is an sequence $\{S_{\sigma,k}(t)\}_k \subset V_0$ such that

$$f(t) = \sum_k f(k+\sigma+\delta_k) S_{\sigma,k}(t) \tag{1}$$

holds for any $f(t) \in V_0$ if

$$\delta_{\sigma,\varphi} < \left(\frac{\|Z_\varphi(\sigma,\omega)G_\varphi(\omega)\|_0 \left\| \dfrac{Z_\varphi(\sigma,\omega)}{G_\varphi(\omega)} \right\|_0}{\|q_\varphi(s,\sigma)\|_{L^{\lambda}_{\sigma}[a,b]}} \right)^{1/\lambda}. \tag{2}$$

Applying the theorem to calculating the B-spline of order 1 scaling function $N_1(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)}$, we find $\delta_{0,N_1} < 1/3\sqrt{3}$ ($\delta_{0,N_1} < 1/2\sqrt{3}$ when $\delta_k \geqq 0$ or $\delta_k \leqq 0$). In this paper we obtain a different estimate. When it is applied to the B-spline or order 1, a relaxed estimate ($\delta_{N_1} < 1/4.94$) can be obtained, and the estimate keeps the same when $\delta_k \geqq 0$ (or $\delta_k \leqq 0$).

Let us now roughly introduce the aforementioned MRA (Multi-resolution Analysis). For more details, see Long-Chen[16] and Long-Chen-Yuan[17] or any books on wavelets, such as Chui[10], Meyer[18] and Walter[22]. An increasing close subspace sequence $\{V_m\}_m$ of $L^2(R)$ is called an MRA if

1. $\cap_m V_m = \{0\}$ and $\cup_m V_m = R$,
2. $f(t) \in V_m$ if and only if $f(2t) \in V_{m+1}$,
3. there is a function $\varphi(t) \in V_0$ (called a scaling function) such that $\{\varphi(t-k)\}_k$ is a Riesz basis of V_0.

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If $\{\varphi(t-k)\}_k$ is an orthonormal Riesz basis of V_0, $\varphi(t)$ is called an orthonormal scaling function. For a scaling function pair $\{\varphi(t), \tilde{\varphi}(t)\}$ of an MRA pair $\{V_m, \tilde{V}_m\}$, if $\int_R \varphi(t-k)\tilde{\varphi}(t-l)dt = \delta_{k,l}$, the MRA pair $\{V_m, \tilde{V}_m\}$ is called a biorthogonal MRA. For an MRA the fact

$$0 < \|G_\varphi(\omega)\|_0 \leqq \|G_\varphi(\omega)\|_\infty < \infty \tag{3}$$

always holds. The $G_\varphi(\omega) = 1$ (a.e.), if $\varphi(t)$ is an orthonormal scaling function.

The following are some notations used in this paper. For a measurable set $E \subset R$, $|E|$ denotes the measure of E. For a measurable function $f(t)$ and a positive number λ, we write

$$\|f\|_0 = \sup_{|E|=0} \inf_{R\setminus E} |f(t)|,$$
$$\|f\|_\infty = \inf_{|E|=0} \sup_{R\setminus E} |f(t)|,$$
$$\hat{f}^*(\omega) = \sum_n f(n)e^{-in\omega},$$
$$G_f(\omega) = \left(\sum_k |\hat{f}(\omega + 2k\pi)|^2\right)^{1/2},$$
$$q_f(s,t) = \sum_n f(s-n)f(t-n),$$
$$L^2(R) = \left\{f \mid \int_R |f(t)|^2 dt < \infty\right\},$$
$$\mathrm{Lip}^\lambda(R) = \{f \mid |f(t) - f(s)| \leqq O(|t-s|^\lambda)\}.$$

Finally we introduce the function class $L_\sigma^\lambda[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$, $0 \in [a,b] \subset [-1,1]$) defined and used in our former works (see Chen-Itoh-Shiki [5] and Chen-Itoh [6]). We have presented some simple properties of the function class in the former works, so we only give the definition here.

Definition 1: A function $f(t) \in L_\sigma^\lambda[a,b]$ ($\lambda > 0, \sigma \in [0,1), 0 \in [a,b] \subset [-1,1]$) if there is a constant $C_{\sigma,f} > 0$ such that for any $\{\delta_k\}_k \subset [a,b]$

$$\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)| \leqq C_{\sigma,f}(\sup_k |\delta_k|)^\lambda. \tag{4}$$

We also write

$$\|f\|_{L_\sigma^\lambda[a,b]} = \sup_{[a,b]} \frac{\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)|}{(\sup_k |\delta_k|)^\lambda}.$$

2. Irregular Sampling in Orthonormal Wavelet Subspaces

Firstly we establish an irregular sampling theorem for orthonormal wavelet subspaces, then we deduce the result we want.

Theorem 2: Let $\varphi(t)$ be an orthonormal scaling function of an MRA $\{V_m\}_m$ such that

1. $\varphi(t) \in L_0^\lambda[a,b]$,
2. $C^{-1} \leqq |\hat{\varphi}^*(\omega)| \leqq C$, $a.e.\,\omega$, for a constant $C \geqq 1$.

Then for any $\{\delta_k\}_k \subset [-\delta_\varphi, \delta_\varphi] \cap [a,b]$, there is a sequence $\{S_k(s)\}_k \subset V_0$ such that

$$f(t) = \sum_k f(k+\delta_k)S_k(t) \tag{5}$$

holds for any $f(t) \in V_0$, if

$$\delta_\varphi < (\|\hat{\varphi}^*(\omega)\|_0 / \|\varphi\|_{L_0^\lambda[a,b]})^{1/\lambda}. \tag{6}$$

Let us recall the following theorem in our former work [6]

Theorem 3: (see Chen-Itoh [6]) Suppose $\{\varphi(t), \tilde{\varphi}(t)\}$ is a scaling function pair of an MRA pair $\{V_m, \tilde{V}_m\}_m$ (with $V_0 = \tilde{V}_0$) such that

1. $\tilde{\varphi}(t) \in L_0^\lambda[a,b]$,
2. $C^{-1} \leqq |\hat{\tilde{\varphi}}^*(\omega)| \leqq C$, $a.e.\,\omega$, for a constant $C \geqq 1$.

Then for any $\{\delta_k\}_k \subset [-\delta_{\varphi,\tilde{\varphi}}, \delta_{\varphi,\tilde{\varphi}}] \cap [a,b]$, there is a sequence $\{S_k(s)\}_k \subset V_0$ such that (5) holds if

$$\delta_\varphi < \left(\frac{\|\hat{\tilde{\varphi}}^*(\omega)G_\varphi(\omega)\|_0}{\|G_\varphi(\omega)\|_\infty \|\tilde{\varphi}\|_{L_0^\lambda[a,b]}}\right)^{1/\lambda}. \tag{7}$$

In the orthonormal case, we know $\varphi(t) = \tilde{\varphi}(t)$ and $G_\varphi(\omega) = 1$ (a.e.). Hence Theorem 3 becomes in fact Theorem 2.

3. Irregular Sampling in General Wavelet Subspaces

Based on the orthonormal case, we can now establish the reconstruction formula for irregularly sampled signals in general wavelet subspace by orthonormalizing the scaling function $\varphi(t)$.

Theorem 4: Suppose a scaling function $\varphi(t)$ of an MRA $\{V_m\}_m$ satisfies

1. $|\varphi(t)| \leqq O(1/|t|^{1+\varepsilon})$ for some $\varepsilon > 0$,
2. $\hat{\varphi}^*(\omega) \neq 0$,
3. $\varphi(t) \in L_0^\lambda[a,b]$.

Then for any $\{\delta_k\}_k \subset [-\delta_\varphi, \delta_\varphi] \cap [a,b]$, there is a sequence $\{S_k(t)\}_k \subset V_0$ such that (5) holds if

$$\delta_\varphi < (\inf |\hat{\varphi}^*(\omega)G_\varphi^{-1}(\omega)| / \|R_\varphi(t)\|_{L_0^\lambda[a,b]})^{1/\lambda}. \tag{8}$$

In order to show the theorem, we need a lemma.

Lemma 1: Suppose a scaling function $\varphi(t)$ of an MRA$\{V_m\}_m$ satisfies $|\varphi(t)| \leqq O(1/|t|^{1+\varepsilon})$ for some $\varepsilon > 0$, then $\hat{\varphi}^*(\omega), G_\varphi(\omega) \in \mathrm{Lip}^\varepsilon$.

Proof Due to $|\varphi(t)| \leqq O(1/|t|^{1+\varepsilon})$, we have

$$\left|\sum_{k\geq n}\varphi(k)e^{in\omega}\right| \leqq \sum_{k\geq n}|\varphi(k)| \leqq O(1/n^{\varepsilon}). \quad (9)$$

Formula (9) implies $\hat{\varphi}^*(\omega) \in \mathrm{Lip}^{\varepsilon}$ (see Edwards[11]). Let $G_{\varphi}^2(\omega) = \sum_k d_k e^{ik\omega}$. Then (see Long-Chen[15] and Meyer[18]),

$$\begin{aligned}|d_k| &= \left|\int_R \varphi(t)\varphi(t+k)dt\right| \\ &\leqq \int_{|t|\leq|k|/2} |\varphi(t)\varphi(t+k)|dt \\ &\quad + \int_{|t|\geq|k|/2} |\varphi(t)\varphi(t+k)|dt \\ &\leqq O(1/|k|^{1+\varepsilon})\int_{|t|\leq|k|/2}|\varphi(t)|dt \\ &\quad + O(1/|k|^{1+\varepsilon})\int_{|t|\geq|k|/2}|\varphi(t+k)|dt \\ &\leqq O(1/|k|^{1+\varepsilon}). \qquad (10)\end{aligned}$$

From (10) we derive $|\sum_{k\geq n} d_k e^{ik\omega}| \leqq O(1/n^{\varepsilon})$. Hence $G_{\varphi}^2(\omega) \in \mathrm{Lip}^{\varepsilon}$. Therefore $G_{\varphi}(\omega) \in \mathrm{Lip}^{\varepsilon}$ due to the 2π-periodicity of $G_{\varphi}(\omega)$.

Proof of theorem Take $R_{\varphi}(t)$ as the Fourier inverse of $\hat{R}_{\varphi}(\omega) = \hat{\varphi}(\omega)G_{\varphi}^{-1}(\omega)$. Then $R_{\varphi}(t)$ is an orthonormal scaling function (see Chui[10], Meyer[18] and Walter[20]). Suppose $G_{\varphi}^2(\omega) = \sum_k d_k e^{ik\omega}$. Then formula (10) implies $\{d_k\}_k \in l^1$. Now suppose $G_{\varphi}^{-1}(\omega) = \sum_k c_k e^{ik\omega}$. Since $G_{\varphi}(\omega) \in \mathrm{Lip}^{\varepsilon}$ and $\|G_{\varphi}(\omega)\|_0 > 0$, we have $\inf G_{\varphi}(\omega) > 0$. Hence $\{c_k\}_k \in l^1$ due to Wiener-Levy Theorem (see pp.178 in Edwards[11]). We can now verify that $R_{\varphi}(t)$ satisfies the two conditions in Theorem 2.

1. For any $\{\delta_k\}_k \subset [a,b]$,

$$\begin{aligned}&\sum_k |R_{\varphi}(k+\delta_k) - R_{\varphi}(k)| \\ &= \sum_k \left|\sum_n c_n(\varphi(k+\delta_k-n) - \varphi(k-n))\right| \qquad (11)\\ &\leqq \sum_n |c_n| \sum_k |\varphi(k+\delta_{k+n}) - \varphi(k)| \qquad (12)\\ &\leqq \sum_n |c_n| \|\varphi\|_{L_0^{\lambda}[a,b]} \sup_k |\delta_k|^{\lambda}. \qquad (13)\end{aligned}$$

Formula (13) implies $R_{\varphi}(t) \in L_0^{\lambda}[a,b]$ (due to $\varphi(t) \in L_0^{\lambda}[a,b]$ and $\{c_k\}_k \in l^1$).

2. On the other hand we have

$$\begin{aligned}\hat{R}_{\varphi}^*(\omega) &= \sum_n e^{-in\omega}\sum_k c_k\varphi(n-k) \qquad (14)\\ &= \sum_k c_k e^{-ik\omega}\sum_n \varphi(n-k)e^{-i(n-k)\omega} \qquad (15)\\ &= G_{\varphi}^{-1}(\omega)\hat{\varphi}^*(\omega). \qquad (16)\end{aligned}$$

The assumption $\hat{\varphi}^*(\omega) \neq 0$ together with the 2π-periodicity implies $0 < \inf|\hat{\varphi}^*(\omega)| \leqq \sup|\hat{\varphi}^*(\omega)| < \infty$. Therefore formula (16) implies

$$0 < \|\hat{R}_{\varphi}^*(\omega)\|_0 \leqq \|\hat{R}_{\varphi}^*(\omega)\|_{\infty} < \infty. \qquad (17)$$

Now we apply Theorem 1 to the scaling function $R_{\varphi}(t)$ of the MRA $\{V_m\}_m$. Then (6) becomes

$$\delta_{\varphi} < (\|\hat{R}_{\varphi}{}^*(\omega)\|_0/\|R_{\varphi}(t)\|_{L_0^{\lambda}[a,b]})^{1/\lambda}. \qquad (18)$$

However, Lemma 1 and (16) imply that $\|\hat{R}_{\varphi}{}^*(\omega)\|_0 = \inf_{\omega}|\hat{\varphi}^*(\omega)G_{\varphi}^{-1}(\omega)|$. Therefore (18) is exactly what we want, i.e.,

$$\delta_{\varphi} < (\inf|\hat{\varphi}^*(\omega)G_{\varphi}^{-1}(\omega)|/\|R_{\varphi}(t)\|_{L_0^{\lambda}[a,b]})^{1/\lambda}. \qquad (19)$$

4. Shift Sampling in Wavelet Subspaces

As done by Janssen[12] for Walter Sampling Theorem[20], Chen-Itoh[6] for irregular sampling, Chen-Itoh[7] for oversampling and Chen-Itoh[9] for regular sampling, the shift version of this irregular sampling theorem also can be obtained by using Zak-transform $Z_{\varphi}(\sigma,\omega)$ ($\sigma \in [0,1)$) defined by

$$Z_{\varphi}(\sigma,\omega) = \sum_n \varphi(\sigma+n)e^{-in\omega}. \qquad (20)$$

Theorem 5: Suppose a scaling function $\varphi(t)$ of an MRA $\{V_m\}_m$ satisfies

1. $|\varphi(t)| \leqq O(1/|t|^{1+\varepsilon})$ for some $\varepsilon > 0$,
2. $Z_{\varphi}(\sigma,\omega) \neq 0$,
3. $\varphi(t) \in L_{\sigma}^{\lambda}[a,b]$.

Then for any $\{\delta_k\}_k \subset [-\delta_{\sigma,\varphi}, \delta_{\sigma,\varphi}] \cap [a,b]$, there is a sequence $\{S_{\sigma,k}(t)\}_k \subset V_0$ such that (1) holds if

$$\delta_{\sigma,\varphi} < \left(\frac{\inf_{\omega}|Z_{\varphi}(\sigma,\omega)G_{\varphi}^{-1}(\omega)|}{\|R_{\varphi}(t)\|_{L_{\sigma}^{\lambda}[a,b]}}\right)^{1/\lambda}. \qquad (21)$$

5. An Example to Show the Algorithm

Example. (see Chui[10]) B-spline of order 1 scaling function $N_1(t) = t\chi_{[0,1)} + (2-t)\chi_{[1,2)}$. Obviously $\hat{N_1}{}^*(\omega) = 1$, and $G_{N_1}(\omega) = (\frac{1}{3} + \frac{2}{3}\cos^2(\frac{\omega}{2}))^{1/2}$. Let $r_k = \frac{1}{2\pi}\int_0^{2\pi}(\frac{1}{3} + \frac{2}{3}\cos^2(\frac{\omega}{2}))^{-1/2}\cos k\omega d\omega$. Then $R_{N_1}(t) = \sum_k r_k N_1(t-k)$. Hence

$$\begin{aligned}&\sum_n |R_{N_1}(n+\delta_n) - R_{N_1}(n)| \\ &= \sum_n\left|\sum_k r_k(N_1(n+\delta_n-k) - N_1(n-k))\right| \qquad (22)\\ &= \sum_n\left|\sum_l r_{n-l}(N_1(l+\delta_n) - N_1(l))\right| \qquad (23)\\ &\leqq \left(\sum_n \max\{|r_n - r_{n-1}|, |r_{n-1} - r_{n-2}|\}\right)\delta_{N_1} \quad (24)\end{aligned}$$

Therefore $\|R_{N_1}(t)\|_{L_0^1[-1,1]} \leqq \sum_n \max(|r_n - r_{n-1}|, |r_{n-1} - r_{n-2}|) < 4.94$. So it is enough to let $\delta_{N_1} < 1/4.94$. Obviously $1/3\sqrt{3} < 1/4.94$. When $\delta_k \geqq 0$ (or $\delta_k \leqq 0$) for all k, $\sum_n |R_{N_1}(n+\delta_n) - R_{N_1}(n)| \leqq (\sum_n |r_n - r_{n-1}|)\delta_{N_1}$. Then $\|R_{N_1}(t)\|_{L_0^1[0,1]} < 3.46411 \approx 2\sqrt{3}$. Therefore $\delta_{N_1} < 1/2\sqrt{3}$.

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