PAPER

On Irregular Sampling in Wavelet Subspaces

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SUMMARY The paper provides the algorithm to estimate the deviation bound admitting to recovering irregularly sampled signals in wavelet subspaces, which does not need the symmetricity sampling constraint of Paley-Wiener's and relaxes the deviation bounds in some wavelet subspaces. Meanwhile the method does not need the continuity and decay constraints imposed on scaling functions by Liu-Walter and Chen-Itoh-Shiki.

key words: *sampling, wavelet, scaling function, orthogonality, biorthogonality, MRA, Zak-transform*

1. Introduction and Notations

For finite energy γ-band continuous signal $f(t)$, $t \in R$. i.e., $f \in L^2(R)$ and $\mathrm{supp}\hat{f}(\omega) = [-\gamma, \gamma]$, the classical Shannon Sampling Theorem gave the following reconstruction formula,

$$f(t) = \sum_n f(nT)\frac{\sin\gamma(t-nT)}{\gamma(t-nT)}, \quad T \leqq \frac{\pi}{\gamma}, \tag{1}$$

where $\hat{f}(\omega)$ is the Fourier transform of $f(t)$ defined by $\hat{f}(\omega) = \int_R f(t)e^{-i\omega t}dt$. Unfortunately it is not appropriate for non-band-limited signals. However if we let $\gamma = 2^m\pi$, $m \in Z$, this problem can be viewed as a special case of wavelet subspaces with $\varphi(t) = \sin\pi t/\pi t$ playing the role of scaling function of MRA $\{V_m = \overline{\mathrm{span}}\{\varphi(2^m t - n\}_n\}_m$. Realizing these properties, Walter[16] extended (1) to a class of wavelet subspaces. Let $\varphi(t)$ be the scaling function of MRA$\{V_m\}_m$ such that $\varphi(t) \leqq O(|t|^{-1-\varepsilon})$ for some $\varepsilon > 0$, which captures many important cases and includes the Haar, Spline and Daubechies Scaling Functions. Walter[16] showed that, in orthonormal case, if $\hat{\varphi}^*(\omega) \neq 0$ there is an $S(t) \in V_0$ such that

$$f(t) = \sum_k f(k)S(t-k) \quad \text{for } f \in V_0. \tag{2}$$

However in many practical cases the sampling are not always at the same step, or say irregular sampling, then how to deal with it? Paley-Wiener's $\frac{1}{4}$-Theorem (see Young[18]) said that, if $\sup_k|\delta_k| < \frac{1}{4}$, $\delta_k = -\delta_{-k}$ then for $f(t) \in P_\pi$ (Paley-Wiener Space),

$$f(t) = \sum_k f(k+\delta_k)\frac{G(t)}{G'(k+\delta_k)G(t-(k+\delta_k))}, \tag{3}$$

where $G(t) = t\prod_{n=1}^{\infty}(1 - t^2/(n+\delta_n)^2)$. But it can not well deal with non-band-limited signals, and the sampling with symmetricity constraints $\delta_k = -\delta_{-k}$ is also restrictive. Following Walter[16], Liu-Walter[12] tried to extend Paley-Wiener's to the sampling in a class of orthonormal wavelet subspaces without $\delta_k = -\delta_{-k}$. But they could not claim the existence of some $\delta_\varphi \subset (0, 1]$ such that a reconstruction formula similar to (3) holds when $\sup_k|\delta_k| < \delta_\varphi$. Then Liu[11] turned to deal with the special case—spline wavelets by Feichtinger-Grochenig Iterative Algorithm (see Feichtinger-Grochenig[8]). Even so, it is to estimate the sampling density, not the deviation bound admitting to recovering original signals. Chen-Itoh-Shiki[2] obtained a recovering formula for sampling in general wavelet subspaces without the symmetricity requirement for sampling but lead to a l^1-bound on $\{\delta_k\}_k$. In fact they still can not estimate the above described δ_φ. Ahthough Chen-Itoh-Shiki[3] obtained an l^∞-bound δ_φ, the continuity assumption and the decay constraint ($\varphi(t) \leqq O(|t|^{-1-\varepsilon})$ for some $\varepsilon > 0$) imposed on scaling function $\varphi(t)$ by Liu-Walter[12] and Chen-Itoh-Shiki[2] can not be removed yet, which even exclude Haar scaling function and Shannon scaling function.

In this paper, we provide the algorithms which can estimate a l^∞ deviation bound to recovering irregularly sampled signals in general wavelet subspaces. It does not require the symmetricity constraints $\delta_k = -\delta_{-k}$ of Paley-Wiener[18]'s for sampling, but also remove the continuity and the decay constraints imposed on the scaling function $\varphi(t)$ by Liu-Walter[12] and Chen-Itoh-Shiki[2]. Furthermore the theorems are modified to a more useful case by using Zak transform (see Janssen[10]). Summarily, we can estimate some $\delta_\varphi \in (0, 1]$, for any $\{\delta_k\}_k$ with $\sup_k|\delta_k| < \delta_\varphi$, there is a $\{S_k(s)\}_k \subset V_0$ such that

$$f(s) = \sum_k f(k+\delta_k)S_k(s) \quad \text{for } f \in V_0. \tag{4}$$

At the end, we calculate some examples and indicate

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that δ_φ can be near 1 when we calculate Haar scaling function, and bigger than $\frac{1}{4}$ of Paley-Wiener [12]'s when we calculate B-Spline of order 1.

We now drop in MRA (Multi-Resolution Analysis) which has been talked about above and can be found in any books on wavelet or see Long-Chen [13] and Long-Chen-Yuan [14]. A subspace sequence $\{V_m\}_m$ of $L^2(R)$ is said to be an MRA if

1. $V_m \subset V_{m+1}$, $\cap_m V_m = \{0\}$, $\overline{\cup_m V_m} = L^2(R)$,
2. Function $f(t) \in V_m$ if and only if $f(2t) \in V_{m+1}$,
3. There is a function (called scaling function) $\varphi(t) \in V_0$ such that $\{\varphi(t-k)\}_k$ forms a Riesz basis in V_0.

The terms Multi Resolution Approximation and Multi Resolution Decomposition sometimes are also used.

MRA$\{V_m\}_m$ is said to be orthogonal (resp. orthonormal) if $\{\varphi(t-k)\}_k$ forms an orthogonal (resp. orthonormal) Riesz basis in V_0. Then the scaling function $\varphi(t)$ is also said to be orthogonal (resp. orthonormal).

MRA pair $\{V_m, \tilde{V}_m\}_m$ is called to be biorthogonal if $\int_R \varphi(t-k)\tilde{\varphi}(t-l)dt = \delta_{kl}$, where $\varphi(t) \in V_0$ and $\tilde{\varphi}(t) \in \tilde{V}_0$ are scaling functions of MRA$\{V_m\}_m$ and $\{\tilde{V}_m\}_m$ respectively. In that case, scaling function pair $\{\varphi(t), \tilde{\varphi}(t)\}$ is also said to be biorthogonal.

The followings are some notations used in this paper. For measurable set $E \subset R$, $|E|$ denotes the measure of E. For measurable function $f(t)$ and $g(t)$ ($t \in R$), $\lambda > 0$, $0 \in [a,b] \subset [-1,1]$, we write

$$\|f\| = \left(\int_R |f(t)|^2\right)^{1/2},$$
$$\|f\|_0 = \sup_{|E|=0} \inf_{R\backslash E} |f(t)|,$$
$$\|f\|_\infty = \inf_{|E|=0} \sup_{R\backslash E} |f(t)|,$$
$$\|f\|_{L^2(E)} = \left(\int_E |f(t)|^2\right)^{1/2},$$
$$L^2(E) = \left\{f(t) : \int_E |f(t)|^2 dt < 0\right\},$$
$$l^1 = \left\{\{a_k\}_k : \sum_k |a_k| < \infty\right\},$$
$$l^2 = \left\{\{a_k\}_k : \sum_k |a_k|^2 < \infty\right\},$$
$$l^\infty = \left\{\{a_k\}_k : \sup_k |a_k| < \infty\right\},$$
$$G_f(\omega) = \left(\sum_k |\hat{f}(\omega + 2k\pi)|^2\right)^{1/2},$$
$$G_{f,g}(\omega) = \left(\sum_k \hat{f}(\omega+2k\pi)\overline{\hat{g}(\omega+2k\pi)}\right)^{1/2},$$
$$q_{f,g}(s,t) = \sum_n f(s-n)g(t-n),$$
$$\hat{f}^*(\omega) = \sum_n f(n)e^{in\omega} = \sum_k \overline{\hat{f}(\omega+2k\pi)},$$
$$\mathrm{Lip}^\lambda = \{f : |f(s) - f(t)| \leqq C_f|s-t|^\lambda,\ s,t \in R\},$$
$$\mathrm{Lip}^\lambda_{[a,b]}(s) = \{f : |f(s+h) - f(s)| \leqq C_f(s)|h|^\lambda,\ h \in [a,b]\},$$
$$\|f\|_{\mathrm{Lip}^\lambda_{[a,b]}(s)} = \sup_{[a,b]} \frac{|f(s+h)-f(s)|}{|h|^\lambda}, \quad \text{for } f(t) \in \mathrm{Lip}^\lambda_{[a,b]}(s).$$

Finally we introduce a function class and display some basic properties of the class. Since the proofs of the propositions are easy we omit them here.

Definition 1: $f(t) \in L^\lambda_\sigma[a,b]$ ($\lambda > 0$, $\sigma \in [0,1)$, $0 \in [a,b] \subset [-1,1]$) holds if there is a constant $C_{\sigma,f} > 0$ such that for any $\{\delta_k\}_k \subset [a,b]$

$$\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)| \leqq C_{\sigma,f}\left(\sup_k |\delta_k|\right)^\lambda. \tag{5}$$

We also write

$$\|f\|_{L^\lambda_\sigma[a,b]} = \sup_{[a,b]} \frac{\sum_k |f(k+\sigma+\delta_k) - f(k+\sigma)|}{(\sup_k |\delta_k|)^\lambda}.$$

Proposition 1:

1. $L^\lambda_\sigma[a,b] = L^\lambda_\sigma[a,0] \cap L^\lambda_\sigma[0,b]$, $L^\lambda_\sigma[a,b] \subset L^{\lambda'}_\sigma[a,b]$ if $\lambda > \lambda'$.
2. $\{f : \sum_k \|f\|_{\mathrm{Lip}^\lambda_{[a,b]}(k+\sigma)} < \infty\} \subset L^\lambda_\sigma[a,b] \subset \cap_k \mathrm{Lip}^\lambda_{[a,b]}(k+\sigma)$.
3. If $f(t)$ is differentiable on each $k+\sigma+[a,b]$ and $\sum_k \sup_{k+\sigma+[a,b]} |f'(t)| < \infty$, then $f(t) \in L^1_\sigma[a,b]$.

2. Irregular Sampling in Biorthogonal Wavelet Subspaces

Firstly we estimate the deviation bound admitting to recovering irregularly sampled signals in biorthogonal wavelet subspaces then deduce that of general case by some tricks (constructing a biorthogonal MRA from a general MRA) in the next section.

Theorem 1: Suppose $\{\varphi(t), \tilde{\varphi}(t)\}$ be a biorthogonal scaling function pair of MRA pair $\{V_m, \tilde{V}_m\}_m$ (with $V_0 = \tilde{V}_0$) satisfying

1. $\tilde{\varphi}(t) \in L^\lambda_0[a,b]$,
2. there is a constant $C \geqq 1$ such that $C^{-1} \leqq |\hat{\tilde{\varphi}}^*(\omega)| \leqq C$ *a.e.* ω.

Then for any $\{\delta_k\}_k \subset [-\delta_{\varphi,\tilde{\varphi}}, \delta_{\varphi,\tilde{\varphi}}] \cap [a,b]$, there is a sequence $\{S_k(s)\}_k$ biorthogonal to $\{q_{\varphi,\tilde{\varphi}}(s, k+\delta_k)\}_k$ in V_0 such that (4) holds if

$$\delta_{\varphi,\tilde{\varphi}} < \left(\frac{\|\hat{\tilde{\varphi}}^*(\omega) G_\varphi(\omega)\|_0}{\|G_\varphi(\omega)\|_\infty \|\tilde{\varphi}\|_{L_0^\lambda[a,b]}} \right)^{1/\lambda}. \tag{6}$$

We need two lemmas for the proof of the Theorem.

Lemma 1: Suppose $\{\varphi(t), \tilde{\varphi}(t)\}$ be a biorthogonal scaling function pair of MRA pair $\{V_m, \tilde{V}_m\}_m$ (with $V_0 = \tilde{V}_0$) satisfying that there is a constant $C \geqq 1$ such that $C^{-1} \leqq |\hat{\tilde{\varphi}}^*(\omega)| \leqq C, a.e.\omega$. Then $\{q_{\varphi,\tilde{\varphi}}(s,k)\}_k$ is a Riesz basis in V_0.

Proof $C^{-1} \leqq |\hat{\tilde{\varphi}}^*(\omega)| \leqq C, a.e.\omega$ assure $\hat{\tilde{\varphi}}^*(\omega) = \sum_k \tilde{\varphi}(k) e^{ik\omega} \in L^2([0,2\pi])$, hence $\{\tilde{\varphi}(k)\}_k \in l^2$. Since $\{\varphi(s-k)\}_k$ is a Riesz basis in V_0, we can be sure that

$$q_{\varphi,\tilde{\varphi}}(s,k) = \sum_n \varphi(s-n)\tilde{\varphi}(k-n)$$

is well-defined and $\{q_{\varphi,\tilde{\varphi}}(s,k)\}_k \subset V_0$. Let T be the linear operator on V_0 that takes $\sum_k c_k \varphi(t-k)$ into $\sum_k c_k q_{\varphi,\tilde{\varphi}}(s,k)$ for any $\{c_k\}_k \in l^2$ (the linearity can be verified easily). Due to Parseval Identity we have

$$\left\| \sum_k c_k q_{\varphi,\tilde{\varphi}}(s,k) \right\|^2$$

$$= \frac{1}{2\pi} \left\| \sum_k c_k \hat{q}_{\varphi,\tilde{\varphi}}(\omega,k) \right\|^2 \tag{7}$$

$$= \frac{1}{2\pi} \left\| \sum_k c_k \sum_n \hat{\varphi}(\omega) e^{-in\omega} \tilde{\varphi}(k-n) \right\|^2 \tag{8}$$

$$= \frac{1}{2\pi} \left\| \sum_k c_k \hat{\varphi}(\omega) \hat{\tilde{\varphi}}^*(\omega) e^{-ik\omega} \right\|^2 \tag{9}$$

$$= \frac{1}{2\pi} \left\| \hat{\tilde{\varphi}}^*(\omega) G_\varphi(\omega) \sum_k c_k e^{-ik\omega} \right\|^2_{L^2[0,2\pi]}. \tag{10}$$

Hence

$$\|\hat{\tilde{\varphi}}^*(\omega) G_\varphi(\omega)\|_0^2 \sum_k |c_k|^2 \leqq \left\| \sum_k c_k q_{\varphi,\tilde{\varphi}}(s,k) \right\|^2 \leqq \|\hat{\tilde{\varphi}}^*(\omega) G_\varphi(\omega)\|_\infty^2 \sum_k |c_k|^2. \tag{11}$$

On the other hand, from

$$\left\| \sum_k c_k \varphi(s-k) \right\|^2 = \frac{1}{2\pi} \left\| \hat{\varphi}(\omega) \sum_k c_k e^{-ik\omega} \right\|^2 \tag{12}$$

$$= \frac{1}{2\pi} \left\| G_\varphi(\omega) \sum_k c_k e^{-ik\omega} \right\|^2_{L^2[0,2\pi]}, \tag{13}$$

we can deduce that

$$\|G_\varphi(\omega)\|_0^2 \sum_k |c_k|^2 \leqq \left\| \sum_k c_k \varphi(s-k) \right\|^2 \leqq \|G_\varphi(\omega)\|_\infty^2 \| \sum_k |c_k|^2. \tag{14}$$

It is well-known (see Chui[1], Meyer[15] and Walter[17])

$$0 < \|G_\varphi(\omega)\|_0 \leqq \|G_\varphi(\omega)\|_\infty < \infty. \tag{15}$$

From (11), (14), (15), now we can conclude that the inverse of T exists (denoted by T^{-1}) and

$$\|T^{-1}\|,\ \|T\| \leqq \frac{C\|G_\varphi(\omega)\|_\infty}{\|G_\varphi(\omega)\|_0} < \infty, \tag{16}$$

i.e., $\{q_{\varphi,\tilde{\varphi}}(s,k)\}_k$ is a Riesz basis in V_0 due to $\{\varphi(t-k)\}_k$ is.

Lemma 2: Suppose the biorthogonal scaling function pair $\{\varphi(t), \tilde{\varphi}(t)\}$ of MRA pair $\{V_m, \tilde{V}_m\}_m$ (with $V_0 = \tilde{V}_0$) satisfy

1. $\tilde{\varphi}(t) \in L_0^\lambda[a,b]$,
2. there is a constant $C \geqq 1$ such that $C^{-1} \leqq |\hat{\tilde{\varphi}}^*(\omega)| \leqq C\ a.e.\omega$.

Then for any $\{\delta_k\}_k \subset [a,b]$,

$$f(k+\delta_k) = \int_R f(s) q_{\varphi,\tilde{\varphi}}(s, k+\delta_k) ds,\ f \in V_0. \tag{17}$$

Proof Due to $\{\tilde{\varphi}(k)\}_k \in l^2$ (refer to the proof of Lemma 1), we have

$$\begin{aligned}
&\sum_n |\tilde{\varphi}(k+\delta_k-n)|^2 \\
&\leqq \sum_n (|\tilde{\varphi}(k-n)| + |\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n)|)^2 \\
&\leqq 2\sum_l \left(|\tilde{\varphi}(l)|^2 + |\tilde{\varphi}(l+\delta_{k+l}) - \tilde{\varphi}(l)|^2\right) \\
&\leqq 2\sum_l |\tilde{\varphi}(l)|^2 + O\left(\sum_l |\tilde{\varphi}(l+\delta_{k+l}) - \tilde{\varphi}(l)|\right) \\
&\leqq 2\sum_l |\tilde{\varphi}(l)|^2 + O\left(\|\tilde{\varphi}\|_{L_0^\lambda[a,b]} \sup_l |\delta_l|^\lambda\right).
\end{aligned} \tag{18}$$

(18) implies $\{\tilde{\varphi}(k+\delta_k-n)\}_n \in l^2$. Therefore

$$q_{\varphi,\tilde{\varphi}}(s, k+\delta_k) = \sum_n \varphi(s-n)\tilde{\varphi}(k+\delta_k-n)$$

is well defined and in V_0. Suppose $f(t) = \sum_k c_k \tilde{\varphi}(t-$

$k) \in \tilde{V}_0 = V_0$, due to the continuity of Fourier transform and Parseval Identity, we have

$$\begin{aligned}
&\int_R f(s) q_{\varphi,\tilde{\varphi}}(s, k+\delta_k) ds \\
&= \frac{1}{2\pi} \int_R \hat{f}(\omega) \overline{\hat{q}_{\varphi,\tilde{\varphi}}(\omega, k+\delta_k)} d\omega && (19)\\
&= \frac{1}{2\pi} \int_R \hat{\tilde{\varphi}}(\omega) \sum_n c_n e^{-in\omega} \\
&\quad \cdot \sum_n \overline{\hat{\varphi}(\omega) \tilde{\varphi}(k+\delta_k-n) e^{-in\omega}} d\omega && (20)\\
&= \frac{1}{2\pi} \int_0^{2\pi} G^2_{\varphi,\tilde{\varphi}}(\omega) \sum_n c_n e^{-in\omega} \\
&\quad \cdot \overline{\sum_n \tilde{\varphi}(k+\delta_k-n) e^{-in\omega}} d\omega && (21)\\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_n c_n e^{-in\omega} \\
&\quad \cdot \overline{\sum_n \tilde{\varphi}(k+\delta_k-n) e^{-in\omega}} d\omega && (22)\\
&= \sum_n c_n \tilde{\varphi}(k+\delta_k-n) && (23)\\
&= f(k+\delta_k), && (24)
\end{aligned}$$

where (22) is due to $G^2_{\varphi,\tilde{\varphi}}(\omega) = 1$, *a.e.* ω (biorthogonality).

Proof of theorem If we can show that $\{q_{\varphi,\tilde{\varphi}}(s, k+\delta_k)\}_k$ is a Riesz basis in V_0, then there is a sequence $\{S_k(s)\}_k$ biorthogonal to $\{q_{\varphi,\tilde{\varphi}}(s, k+\delta_k)\}_k$ in V_0, such that

$$f(s) = \sum_k S_k(s) \int_R f(s) q_{\varphi,\tilde{\varphi}}(s, k+\delta_k) ds, \ f \in V_0. \tag{25}$$

Following Lemma 2, it is easy to see (4) holds. However Lemma 1 tells us that $\{q_{\varphi,\tilde{\varphi}}(s,k)\}_k$ is a Riesz basis in V_0. So we only need to show $\{q_{\varphi,\tilde{\varphi}}(s, k+\delta_k)\}_k$ is an equivalent basis of $\{q_{\varphi,\tilde{\varphi}}(s,k)\}_k$ in V_0, i.e., to find a $\delta_{\varphi,\tilde{\varphi}} \in (0,1]$, for any $\{\delta_k\}_k \subset [-\delta_{\varphi,\tilde{\varphi}}, \delta_{\varphi,\tilde{\varphi}}] \cap [a,b]$, there is a $\theta \in [0,1)$ such that for any $\{c_k\}_k \in l^2$,

$$\begin{aligned}
&\left\| \sum_k c_k q_{\varphi,\tilde{\varphi}}(s, k+\delta_k) - \sum_k c_k q_{\varphi,\tilde{\varphi}}(s,k) \right\|^2 \\
&\leq \theta \left\| \sum_k c_k q_{\varphi,\tilde{\varphi}}(s,k) \right\|^2 . && (26)
\end{aligned}$$

In order to show (26), let

$$\begin{aligned}
\Delta &= \left\| \sum_k c_k q_{\varphi,\tilde{\varphi}}(s, k+\delta_k) - \sum_k c_k q_{\varphi,\tilde{\varphi}}(s,k) \right\|^2 \\
&= \left\| \sum_n \left(\sum_k c_k (\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n)) \right) \varphi(s-n) \right\|^2 \\
&= \frac{1}{2\pi} \left\| \sum_n \left(\sum_k c_k (\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n)) \right) \hat{\varphi}(\omega) e^{-in\omega} \right\|^2 \\
&= \frac{1}{2\pi} \left\| G_\varphi(\omega) \sum_n \left(\sum_k c_k (\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n)) \right) e^{-in\omega} \right\|^2_{L^2[0,2\pi]} \\
&\leq \|G_\varphi(\omega)\|^2_\infty \sum_n \left| \sum_k c_k (\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n)) \right|^2 \\
&= \|G_\varphi(\omega)\|^2_\infty \sum_n \sum_{k,l} (\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n))(\tilde{\varphi}(l+\delta_l-n) - \tilde{\varphi}(l-n)) c_k c_l \\
&= \|G_\varphi(\omega)\|^2_\infty \sum_{k,l} \left(\sum_n (\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n))(\tilde{\varphi}(l+\delta_l-n) - \tilde{\varphi}(l-n)) \right) c_k c_l .
\end{aligned}$$

Take

$$\begin{aligned}
b_{k,l} &= \sum_n (\tilde{\varphi}(k+\delta_k-n) - \tilde{\varphi}(k-n)) \\
&\quad \cdot (\tilde{\varphi}(l+\delta_l-n) - \tilde{\varphi}(l-n)). && (27)
\end{aligned}$$

Then $b_{k,l} = b_{l,k}$ and

$$\begin{aligned}
\Delta &\leq \|G_\varphi(\omega)\|^2_\infty \sum_{k,l} b_{k,l} c_k c_l && (28)\\
&\leq \|G_\varphi(\omega)\|^2_\infty \sum_{k,l} |b_{k,l}| (c_k^2 + c_l^2)/2 && (29)\\
&= \frac{1}{2} \|G_\varphi(\omega)\|^2_\infty \left(\sum_k \left(\sum_l |b_{k,l}| \right) c_k^2 \right. \\
&\quad \left. + \sum_l \left(\sum_k |b_{k,l}| \right) c_l^2 \right) && (30)\\
&= \|G_\varphi(\omega)\|^2_\infty \sum_k \left(\sum_l |b_{k,l}| \right) c_k^2 && (31)\\
&\leq \|G_\varphi(\omega)\|^2_\infty \left(\sup_k \sum_l |b_{k,l}| \right) \sum_k c_k^2 . && (32)
\end{aligned}$$

Meanwhile

$$\sup_k \sum_l |b_{k,l}|$$

$$\leqq \sup_k \sum_l \sum_n |\tilde{\varphi}(k+\delta_k-n)-\tilde{\varphi}(k-n)| \cdot |\tilde{\varphi}(l+\delta_l-n)-\tilde{\varphi}(l-n)|$$
$$\leqq \sup_k \sum_\alpha |\tilde{\varphi}(\alpha+\delta_k)-\tilde{\varphi}(\alpha)| \cdot \sum_\beta |\tilde{\varphi}(\beta+\delta_{\beta+k-\alpha})-\tilde{\varphi}(\beta)|, \tag{33}$$

where (33) is due to the index transform $\alpha = k-n$ and $\beta = l-n$. From $\tilde{\varphi}(t) \in L_0^\lambda[a,b]$, we know that

$$\sum_\alpha |\tilde{\varphi}(\alpha+\delta_k)-\tilde{\varphi}(\alpha)| \sum_\beta |\tilde{\varphi}(\beta+\delta_{\beta+k-\alpha})-\tilde{\varphi}(\beta)|$$
$$\leqq \|\tilde{\varphi}\|^2_{L_0^\lambda[a,b]} \sup_k |\delta_k|^\lambda \sup_k |\delta_{\beta+k-\alpha}|^\lambda \tag{34}$$
$$\leqq \left(\delta^\lambda_{\varphi,\tilde{\varphi}} \|\tilde{\varphi}\|_{L_0^\lambda[a,b]}\right)^2. \tag{35}$$

On the other hand

$$\left\|\sum_k c_k q_{\varphi,\tilde{\varphi}}(s,k)\right\|^2$$
$$= \frac{1}{2\pi}\left\|\sum_k c_k \hat{q}_{\varphi,\tilde{\varphi}}(\omega,k)\right\|^2 \tag{36}$$
$$= \frac{1}{2\pi}\left\|\sum_k c_k \hat{\varphi}(\omega)\sum_n e^{-in\omega}\tilde{\varphi}(k-n)\right\|^2 \tag{37}$$
$$= \frac{1}{2\pi}\left\|\hat{\varphi}(\omega)\sum_k c_k \hat{\tilde{\varphi}}^*(\omega)e^{-ikw}\right\|^2 \tag{38}$$
$$= \frac{1}{2\pi}\left\|\hat{\tilde{\varphi}}^*(\omega)G_\varphi(\omega)\sum_k c_k e^{-ikw}\right\|^2_{L^2[0,2\pi]} \tag{39}$$
$$\geqq \frac{1}{2\pi}\|\hat{\tilde{\varphi}}^*(\omega)G_\varphi(\omega)\|_0^2 \left\|\sum_k c_k e^{-ikw}\right\|^2_{L^2[0,2\pi]} \tag{40}$$
$$= \|\hat{\tilde{\varphi}}^*(\omega)G_\varphi(\omega)\|_0^2 \sum_k |c_k|^2. \tag{41}$$

(26), (32), (33), (35), (41) imply that we only need

$$\|G_\varphi(\omega)\|^2_\infty \left(\delta^\lambda_{\varphi,\tilde{\varphi}} \|\tilde{\varphi}\|_{L_0^\lambda[a,b]}\right)^2 < \|\hat{\tilde{\varphi}}^*(\omega)G_\varphi(\omega)\|_0^2. \tag{42}$$

It is nothing but (6).

Remark 1

1. In the theorem we do not only obtain a L^∞-bound for $\{\delta_k\}_k$ but also remove the continuity and decay constraints imposed on scaling functions by Liu-Walter [12] and Chen-Itoh-Shiki [2].
2. $\{S_k(t)\}_k$ can be calculated as biorthogonal to $\{q_{\varphi,\tilde{\varphi}}(t,k)\}_k$.
3. From Proposition 1, we know that $L_0^\lambda[a,b]$ is a mind collection, which captures many important cases and includes any local Lip-continuous compactly supported scaling functions such as Haar, Spline and Daubechies scaling functions.
4. $\|\tilde{\varphi}(t)\|_{L_0^\lambda[a,b]}$ in (6) can be zero. Then Theorem 1 holds for any irregular sampling with deviations $\{\delta_k\}_k \subset [a,b]$ (refer to Example 1 in Sect. 5).
5. When $\varphi(t)$ is an orthogonal scaling function, $G_\varphi(\omega)$ is constant $a.e.\omega$ and $\tilde{\varphi}(t) = \varphi(t)/\|\varphi\|^2$. Then (6) becomes to be $(\|\varphi\|^{-1}_{L_0^\lambda[a,b]}\|\hat{\varphi}^*(\omega)\|_0)^{1/\lambda}$.
6. In the end we will calculate some examples to reason that δ_φ can be bigger than $\frac{1}{4}$ of Paley-Wiener [18]'s.

3. Irregular Sampling in General Wavelet Subspaces

Now following the results for biorthogonal wavelet subspaces, we can provide an algorithm for irregular sampling in general wavelet subspaces.

Theorem 2: Suppose the scaling function $\varphi(t)$ of MRA $\{V_m\}_m$ satisfy that

1. $\varphi(t) \in L_0^\lambda[a,b]$.
2. $\{\varphi(k)\}_k \in l^1$.
3. There is a constant $C \geqq 1$ such that $C^{-1} \leqq |\hat{\varphi}^*(\omega)| \leqq C$, $a.e.\ \omega$.

Then there is a $\delta_\varphi \in (0,1]$, for any $\{\delta_k\}_k \subset [-\delta_\varphi, \delta_\varphi] \cap [a,b]$, there is a $\{S_k(t)\}_k \subset V_0$ such that (4) holds if

$$\delta_\varphi < \left(\frac{\|\hat{\varphi}^*(\omega)/G_\varphi(\omega)\|_0 \|\hat{\varphi}^*(\omega)G_\varphi(\omega)\|_0}{\|q_\varphi(s,0)\|_{L_0^\lambda[a,b]}}\right)^{1/\lambda}. \tag{43}$$

In order to show the theorem, we need a lemma.

Lemma 3: Suppose the scaling function $\varphi(t)$ of MRA$\{V_m\}_m$ satisfy that there is a constant $C \geqq 1$ such that $C^{-1} \leqq |\hat{\varphi}^*(\omega)| \leqq C$, $a.e.\ \omega$. Then $\{q_\varphi(t,k)\}_k$ is a Riesz basis in V_0. Suppose $\{\tilde{q}_k(t)\}_k$ be biorthogonal to $\{q_\varphi(t,k)\}_k$. Then $\tilde{q}_k(t) = \tilde{q}_0(t-k)$, $\hat{\tilde{q}}_0(\omega) = \hat{\varphi}(\omega)/\overline{\hat{\varphi}^*(\omega)}G^2_\varphi(\omega)$.

Proof Referring to Lemma 1, it is easy to show that $\{q_\varphi(t,k)\}_k$ is a Riesz basis in V_0. Since $q_\varphi(t,k) = q_\varphi(t-k,0)$, and $\{\tilde{q}_0(t-k)\}_k$ is biorthogonal to $q_\varphi(t-k,0)$, we have $\tilde{q}_k(t) = \tilde{q}_0(t-k)$ due to the uniqueness of $\tilde{q}_k(t)$ as biorthogonal to $\{q_\varphi(t,k)\}_k$ (see Daubechies [5], Daubechies-Grossman-Meyer [6], Walter [17] or Young [18]). Suppose $1/\overline{\hat{\varphi}^*}G^2_\varphi(\omega) = \sum_k c_k e^{ik\omega}$. Since $1/\overline{\hat{\varphi}^*}G^2_\varphi(\omega) \in L^\infty[0,2\pi] \subset L^2[0,2\pi]$, we derive that $\{c_k\}_k \in l^2$ holds. Since $\|\frac{\hat{\varphi}(\omega)}{\overline{\hat{\varphi}^*(\omega)}G^2_\varphi(\omega)}e^{-il\omega}\| \leqq O(\|\varphi\|)$, we can take the inverse

Fourier transform of $\frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)G_\varphi^2(\omega)}e^{-il\omega}$ in $L^2(R)$ as denoted by $[\frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)G_\varphi^2(\omega)}e^{-il\omega}]^\vee(s)$ (refer to the Introduction). The above arguments show that

$$\left[\frac{\hat{\varphi}(\omega)}{\overline{\hat{\varphi}^*(\omega)}G_\varphi^2(\omega)}e^{-il\omega}\right]^\vee(s) = \sum_k c_k\varphi(s+k-l). \tag{44}$$

(44) implies $[\frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)G_\varphi^2(\omega)}e^{-il\omega}]^\vee(s) \in V_0$. By the way, since

$$\begin{aligned}
&\int_R q_\varphi(s,k)\overline{\left[\frac{\hat{\varphi}(\omega)}{\overline{\hat{\varphi}^*(\omega)}G_\varphi^2(\omega)}e^{-il\omega}\right]^\vee(s)}ds \\
&= \frac{1}{2\pi}\int_R \hat{q}_\varphi(\omega,k)\frac{\overline{\hat{\varphi}(\omega)}}{\hat{\varphi}^*(\omega)G_\varphi^2(\omega)}e^{il\omega}d\omega \\
&= \frac{1}{2\pi}\int_R \hat{\varphi}^*(\omega)\hat{\varphi}(\omega)e^{-ik\omega}\frac{\overline{\hat{\varphi}(\omega)}}{\hat{\varphi}^*(\omega)G_\varphi^2(\omega)}e^{il\omega}d\omega \\
&= \frac{1}{2\pi}\int_R \frac{|\hat{\varphi}(\omega)|^2}{G_\varphi^2(\omega)}e^{-i(k-l)\omega}d\omega &(45)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sum_k|\hat{\varphi}(\omega+2k\pi)|^2}{G_\varphi^2(\omega)}e^{-i(k-l)\omega}d\omega &(46)\\
&= \delta_{kl}, &(47)
\end{aligned}$$

(47) implies that $\{[\frac{\hat{\varphi}(\omega)}{\hat{\varphi}^*(\omega)G_\varphi^2(\omega)}e^{-ik\omega}]^\vee(s)\}_k$ is biorthogonal to $\{q_\varphi(s,k)\}_k$. Hence

$$\tilde{q}_0(s-k) = \left[\frac{\hat{\varphi}(\omega)}{\overline{\hat{\varphi}^*(\omega)}G_\varphi^2(\omega)}e^{-ik\omega}\right]^\vee(s)$$

due to the uniqueness of $\{\tilde{q}_0(s-k)\}_k$ in V_0. It deduces that

$$\hat{\tilde{q}}_0(\omega) = \frac{\hat{\varphi}(\omega)}{\overline{\hat{\varphi}^*(\omega)}G_\varphi^2(\omega)}. \tag{48}$$

Proof of theorem We have, for any $\{\delta_k\}_k \subset [a,b]$,

$$\begin{aligned}
&\sum_k |q_\varphi(k+\delta_k,0) - q_\varphi(k,0)| \\
&= \sum_k \left|\sum_n \varphi(k+\delta_k-n)\varphi(-n) - \sum_n \varphi(k-n)\varphi(-n)\right| \\
&\leqq \sum_n |\varphi(-n)| \sum_k |\varphi(k+\delta_k-n) - \varphi(k-n)| \\
&= \sum_n |\varphi(-n)| \sum_l |\varphi(l+\delta_{l+n}) - \varphi(l)| \\
&\leqq \sum_n |\varphi(-n)| \|\varphi\|_{L_0^\lambda[a,b]} \sup_k |\delta_k|^\lambda.
\end{aligned}$$

This implies $q_\varphi(s,0) \in L_0^\lambda[a,b]$ due to $\{\varphi(n)\}_n \in l^1$. On the other hand since

$$\hat{q}_\varphi^*(\omega,0) = \hat{\varphi}^*(\omega)\overline{\hat{\varphi}^*}(\omega), \tag{49}$$

we conclude $C^{-2} \leqq |\hat{q}_\varphi^*(\omega,0)| \leqq C^2$. Hence we can apply theorem 1 to the biorthogonal pair $\{\tilde{q}_0(t), q_\varphi(t,0)\}$, i.e., we only need

$$\delta_\varphi < \left(\frac{\|\hat{q}_\varphi^*(\omega,0)G_{\tilde{q}_0}(\omega)\|_0}{\|G_{\tilde{q}_0}(\omega)\|_\infty \|q_\varphi(s,0)\|_{L_0^\lambda[a,b]}}\right)^{1/\lambda}. \tag{50}$$

Since

$$\begin{aligned}
G_{\tilde{q}_0}(\omega) &= \left(\sum_k |\hat{\tilde{q}}_0(\omega+2k\pi)|^2\right)^{1/2} &(51)\\
&= \left(\sum_k \left|\hat{\varphi}(\omega+2k\pi)/\overline{\hat{\varphi}^*(\omega)}G_\varphi^2(\omega)\right|^2\right)^{1/2} &(52)\\
&= G_\varphi(\omega)/\left|\overline{\hat{\varphi}^*(\omega)}G_\varphi^2(\omega)\right| &(53)\\
&= 1/|\hat{\varphi}^*(\omega)|G_\varphi(\omega), &(54)
\end{aligned}$$

(50) becomes to be (43).

Remark 2

1. We do not only obtain a $L^\infty-$ bound for $\{\delta_k\}_k$ but can also remove the continuity and decay constraints imposed on scaling function $\varphi(t)$ by Liu-Walter [12] and Chen-Itoh-Shiki [2], but the alternative weaker condition $\{\varphi(n)\}_n \in l^1$ should be assumed.
2. The $\{S_k(s)\}_k$ in Theorem 2 is biorthogonal to $\{q_{\tilde{q}_0(\cdot),q_\varphi(\cdot,0)}(s,k+\delta_k)\}_k$.
3. $\|q_\varphi(t,0)\|_{L_0^\lambda[a,b]}$ in (43) can be 0. Then Theorem 2 holds for any irregular sampling with deviation $\{\delta_k\}_k \subset [a,b]$.
4. In orthogonal wavelet subspaces, $G_\varphi(\omega)$ is a constant, then (43) becomes $\delta_\varphi < (\frac{\|\hat{\varphi}^*(\omega)\|_0^2}{\|q_\varphi(t,0)\|_{L_0^\lambda[a,b]}})^{1/\lambda}$.
5. When $\varphi(t)$ is a cardinal scaling function (see Walter [17]), $q_\varphi(t,0) = \varphi(t), \hat{\varphi}^*(\omega) = 1$ $a.e.\omega$. Therefore (43) is $\delta_\varphi < (\frac{\|G_\varphi^{-1}(\omega)\|_0\|G_\varphi(\omega)\|_0}{\|\varphi\|_{L_0^\lambda[a,b]}})^{1/\lambda}$.

4. Modified Theorem for General Wavelet Subspaces

If there is a constant $C \geqq 1$ such that $C^{-1} \leqq |\hat{\varphi}^*(\omega)| \leqq C$, $a.e.$ ω, we can apply the above algorithms to deal with the irregularly sampled signals. Unfortunately some scaling functions, even some important scaling functions, do not show the property. For example, take the B-spline of order 2 scaling function

$$N_2(t) = \frac{x^2}{2}\chi_{[0,1)}(t) + \frac{6t-2t^2-3}{2}\chi_{[1,2)}(t) + \frac{(3-t)^2}{2}\chi_{[2,3)}(t), \tag{55}$$

where $\chi_{[j,j+1)}(t)$ is the characteristic function of the interval $[j, j+1)$ for $j = 0, 1, 2$. Then $\hat{N_2}^*(\omega) = \frac{1}{2}e^{i\omega}(e^{i\omega}+1) = 0$ when $\omega = \pi$. So we should find a proper way to solve it. This is the main purpose of the section.

Suppose the scaling function $\varphi(t)$ satisfy $\{\varphi(n+\sigma)\}_n \in l^2$ for some $\sigma \in [0,1)$. Then we can define Zak-transform of φ (see Heil-Walnut [9], Janssen [10] or Walter [17]) as

$$Z_\varphi(\sigma,\omega) = \sum_n \varphi(\sigma+n)e^{in\omega}, \quad \omega \in R. \tag{56}$$

For the above B-spline of order 2 scaling function $N_2(t)$, we find

$$Z_{N_2}\left(\frac{1}{2},\omega\right) = (1+6e^{i\omega}+e^{2i\omega})/8 \neq 0.$$

This implies that we can improve our above algorithms by sampling at $\{\sigma+k\}$ instead of $\{k\}$ for some $\sigma \in [0,1)$. Firstly we modify the theorem and algorithm for irregular sampling in biorthogonal wavelet subspaces, then deduce the modified results for general case. Since the procedure and trick are similar to the previous sections except that $Z_\varphi(\sigma,\omega)$ takes the role of $\hat{\varphi}^*(\omega)$, here we will not show it in detail. Now we only display the results as what follows.

Theorem 3: Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}_m$ and for some $\sigma \in [0,1)$, satisfy

1. $\{\varphi(k+\sigma)\}_k \in l^1$.
2. $\varphi(t) \in L^\lambda_\sigma[a,b]$.
3. There is a constant $C \geqq 1$ such that $C^{-1} \leqq |Z_\varphi(\sigma,\omega)| \leqq C$, *a.e.* ω.

Then there is a $\delta_{\sigma,\varphi} \in (0,1]$ such that for any $\{\delta_k\}_k \subset [-\delta_{\sigma,\varphi}, \delta_{\sigma,\varphi}] \cap [a,b]$, there exists a $\{S_{\sigma,k}(t)\}_k \subset V_0$ such that

$$f(t) = \sum_k f(k+\delta_k+\sigma)S_{\sigma,k}(t) \quad \text{for } f(t) \in V_0 \tag{57}$$

holds if

$$\delta_{\sigma,\varphi} < \left(\frac{\|Z_\varphi(\sigma,\omega)G_\varphi(\omega)\|_0 \left\|\frac{Z_\varphi(\sigma,\omega)}{G_\varphi(\omega)}\right\|_0}{\|q_\varphi(s,\sigma)\|_{L^\lambda_\sigma[a,b]}}\right)^{1/\lambda}. \tag{58}$$

Remark 3.

1. The $\{S_{\sigma,k}(s)\}_k$ in Theorem 3 is biorthogonal to $\{q_{\tilde{q}_{\sigma,0}(\cdot),q_\varphi(\cdot,\sigma)}(s, k+\sigma+\delta_k)\}_k$, where $\tilde{q}_{\sigma,k}(t) = \tilde{q}_{\sigma,0}(t-k)$ is biorthogonal to $q_\varphi(t, k+\sigma)$.
2. Since $Z_\varphi(0,\omega) = \hat{\varphi}^*(\omega)$, (58) is the same to (43) when $\sigma = 0$.
3. When $\|q_\varphi(s,\sigma)\|_{L^\lambda_\sigma[a,b]} = 0$, (57) holds for any irregular sampling with deviation $\{\delta_k\}_k \subset [a,b]$.

5. Conclusion and Examples

1. Suppose $\{V_m\}_m$ be a Multi Resolution Decomposition of $L^2(R)$ with the scaling function $\varphi(t)$ satisfying, for some $\sigma \in [0,1)$,

   A. $\{\varphi(k+\sigma)\}_k \in l^1$.
   B. $C^{-1} \leqq |Z_\varphi(\sigma,\omega)| \leqq C$ *a.e.* ω. for some $C \geqq 1$.
   C. $\varphi(t) \in L^\lambda_\sigma[a,b]$, $(\lambda > 0,\ 0 \in [a,b] \subset [-1,1])$.

   We can assure that there is a $\delta_{\sigma,\varphi} \in (0,1]$, for any irregularly sampled signals $\{f(k+\delta_k)\}_k$ with $\sup_k|\delta_k| < \delta_{\sigma,\varphi}$, the original signal can be reconstructed via

   $$f(s) = \sum_k f(k+\sigma+\delta_k)S_{\sigma,k}(s), \ f \in V_0, \tag{59}$$

   if

   $$\delta_{\sigma,\varphi} < \left(\frac{\|Z_\varphi(\sigma,\omega)G_\varphi(\omega)\|_0 \left\|\frac{Z_\varphi(\sigma,\omega)}{G_\varphi(\omega)}\right\|_0}{\|q_\varphi(s,\sigma)\|_{L^\lambda_\sigma[a,b]}}\right)^{1/\lambda}. \tag{60}$$

2. When $\|q_\varphi(s,\sigma)\|_{L^\lambda_\sigma[a,b]} = 0$, (59) holds for any $\{\delta_k\}_k \subset [a,b]$.
3. If the sampling step is not at 1, or say $T = 2^{-m}$, we can regard V_m as V_0. All the theorems and algorithms can be modified to V_m easily by using Hilbert Reproducing Kernel $q^{(m)}_{\varphi,\tilde{\varphi}}(s,t) = 2^m\sum_n \varphi(2^m s-n)\tilde{\varphi}(2^m t-n)$.
4. In fact we have not used the dilation equation, therefore all the theorems are correct only under the hypothesis that $\{\varphi(t-n)\}_n$ is a Riesz basis of V_0 instead of that $\varphi(t)$ is a scaling function, i.e., only assume $0 < \|G_\varphi(\omega)\|_0 \leqq \|G_\varphi(\omega)\|_\infty < \infty$.

Now we apply the algorithm to calculate some examples.

Example 1. (see Walter [17]) Haar scaling function $\varphi(t) = \chi_{[0,1)}(t)$. Obviously it satisfies **A**. For any $\sigma \in [0,1)$, $Z_\varphi(\sigma,\omega) = 1 \neq 0$. It satisfies **C** with $\sigma \in [0,1)$, $\lambda = 1$, $[a,b] = [-\sigma, 1-\sigma)$. Since $\|\varphi\|_{L^1_\sigma(-\sigma,1-\sigma)} = 0$, from Conclusion 2, we know that (57) holds for any $\{\delta_k\}_k \subset [\sigma, 1-\sigma)$.

Example 2. (see Daubechies [4]) Daubechies scaling function $\varphi_N(t)$ $(N = 1, 2, 3, \cdots)$ is defined as $\hat{\varphi}_N(\omega) = \prod_1^\infty H(2^{-n}\omega)$, where $H(\omega) = (\frac{1+e^{-i\omega}}{2})^N M_N(\omega)$ and $M_N(\omega) = \sum_0^{N-1} C_{N-1+n}^n (\sin^2 \frac{\omega}{2})^n$. It has been shown that $\varphi_N(t)$ is orthonormal, $\mathrm{supp}\varphi_N \subset [0, 2N-1]$ and $\varphi_N(t) \in Lip^{(\min\{\mu N, 1\})}$, $\mu = 0.18$. Therefore $\varphi_N(t) \in L_\sigma^{(\min\{\mu N,1\})}[-1, 1]$ for any $\sigma \in [0, 1)$ due to Proposition 1. If $Z_{\varphi_N}(\sigma, \omega) \neq 0$ for some $\sigma \in [0, 1)$, then

$$\delta_{\sigma,\varphi_N} < \left(\frac{\inf_\omega |Z_{\varphi_N}(\sigma, \omega)|}{2N\|\varphi_N\|_{Lip^{(\min\{\mu N,1\})}}} \right)^{1/(\min\{\mu N,1\})}. \tag{61}$$

Example 3. (see Meyer [15]) Meyer scaling function is defined as

$$\hat{\varphi}(\omega) = \begin{cases} 1 & |\omega| \leqq 2\pi/3, \\ \cos\left[\frac{\pi}{2} v\left(\frac{3}{2\pi}|\omega| - 1\right)\right] & \\ & 2\pi/3 \leqq |\omega| \leqq 4\pi/3, \\ 0 & \text{otherwise,} \end{cases}$$

where $v(\omega) \in C^\infty$, $v(\omega) = 1$ when $\omega \geqq 1$, $v(\omega) = 0$ when $\omega \leqq 0$ and $v(\omega) + v(1 - \omega) = 1$. It is shown that $\varphi(t)$ is orthonormal and r-regular. therefore $\sum_n \sup_{[n,n+1]} |\varphi'(t)|$ converges, hence $\varphi(t) \in L_\sigma^1[-1, 1]$ for any $\sigma \in [0, 1)$ due to Proposition 1. Since

$$\begin{aligned} &\inf |\hat{\varphi}^*(\omega)| \\ &= \min \left\{ 1, \inf_{\frac{-4\pi}{3} \leqq \omega \leqq \frac{-2\pi}{3}} \left| \cos\left(\frac{\pi}{2} v\left(\frac{3\omega}{2\pi} + 2\right)\right) \right. \right. \\ &\quad \left. \left. + \cos\left(\frac{\pi}{2} v\left(-\frac{3\omega}{2\pi} - 1\right)\right) \right| \right\} \\ &= 1, \end{aligned}$$

we obtain

$$\delta_\varphi < \left(\sum_n \sup_{[n,n+1]} |\varphi'(t)| \right)^{-1}. \tag{62}$$

The following example indicates that δ_φ can be bigger than $\frac{1}{4}$ of Paley-Wiener's for some wavelet subspaces.

Example 4. (see Chui [1]) the B-spline of order 1 scaling function $N_1(t) = t\chi_{[0,1)} + (2 - t)\chi_{[1,2)}$. Obviously $N_1(t)$ satisfies constraint **A** and **C** (with $\sigma = 0, \lambda = 1$). Since $\hat{N_1}^*(\omega) = 1$, $G_{N_1}(\omega) = (\frac{1}{3} + \frac{2}{3}\cos^2(\frac{\omega}{2}))^{1/2}$, $\sum_k N_1(t + k)N_1(k) = N_1(t)$, therefore (60) becomes

$$\begin{aligned} \delta_\varphi < &\left\| \left(\frac{1}{3} + \frac{2}{3}\cos^2\left(\frac{\omega}{2}\right) \right)^{1/2} \right\|_0 \\ &\cdot \left\| \frac{1}{\left(\frac{1}{3} + \frac{2}{3}\cos^2\left(\frac{\omega}{2}\right)\right)^{1/2}} \right\|_0 \|N_1\|_{L_0^1[a,b]}^{-1}. \end{aligned} \tag{63}$$

Since $\|N_1\|_{L_0^1[-1,1]} = 3$, $\|N_1\|_{L_0^1[-1,0]} = 2$ and $\|N_1\|_{L_0^1[0,1]} = 2$, therefore $\delta_\varphi < \frac{1}{3\sqrt{3}}$. When $\delta_k \leqq 0$ for all k or $\delta_k \geqq 0$ for all k, then $\delta_\varphi < \frac{1}{2\sqrt{3}}$. Obviously $\frac{1}{2\sqrt{3}} > \frac{1}{4}$.

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