LETTER

Some Notes on Reconstructing Regularly Sampled Signal by Scaling Function with Oversampling Property

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SUMMARY The fact that bounded interval band orthonormal scaling function shows oversampling property is demonstrated. The truncation error is estimated when scaling function with oversampling property is used to recover signals from their discrete samples.
key words: scaling function, oversampling property, interval band, truncation error

1. Introduction

Shannon Sampling Theorem plays a very important role in signal processing. Realizing that Shannon function $\sin t/t$ is in fact an orthonormal scaling function of an MRA, Walter [12] established a sampling theorem for a class of wavelet subspaces.

Suppose $\varphi(t)$ is a continuous orthonormal scaling function of an MRA $\{V_m\}_{m \in Z}$ such that $|\varphi(t)| \leq O(|t|^{-1-\varepsilon})$ for some $\varepsilon > 0$ when $|t| \to \infty$. Let $\hat{\varphi}^*(\omega) = \sum_n \varphi(n) e^{-in\omega}$ Walter showed that there is an $S(t) \in V_0$ such that

$$f(t) = \sum_{n \in Z} f(n) S(t-n) \quad \text{for} \quad f \in V_0 \tag{1}$$

holds for any $f(t) \in V_0$ if $\hat{\varphi}^*(\omega) \neq 0$. Following Walter's work Janssen [7] studied the shift sampling in wavelet subspaces. Chen-Itoh [2] found a sufficient-necessary condition for (1) in shift invariant subspaces. Liu-Walter [9], Liu [8], Chen-Itoh-Shiki [4], [5] and Chen-Itoh [1] even extended it to the irregular sampling in wavelet subspaces. Meanwhile, Xia-Zhang [16] discussed the so called sampling property, i.e.,

$$f(t) = \sum_n f(n) \varphi(t-n) \quad \text{for} \quad f \in V_0. \tag{2}$$

It is also shown that $\varphi(t)$ shows sampling property if and only if $\hat{\varphi}^*(\omega) = 1$. Obviously many important scaling functions such as Haar scaling function, Shannon scaling function and B-spline of order 1 show sampling property. However there are also many important scaling functions, such as certain of Meyer type scaling function and Daubechies scaling function, do not show

sampling property. Thus Walter [13] proposed a weaker expression as

$$f(t) = \sum_n f\left(\frac{n}{2}\right) \varphi(2t-n) \quad \text{for} \quad f \in V_0. \tag{3}$$

(3) is the so-called oversampling property. Xia [15] extended (3) to a more general form as that for some $J \in Z^+ \cup \{0\}$,

$$f(t) = \sum_n f\left(\frac{n}{2^J}\right) \varphi(2^J t - n) \quad \text{for} \quad f \in V_0. \tag{4}$$

(4) is the so-called oversampling property with rate J. It is shown that $\varphi(t)$ satisfies oversampling property with rate J ($J \in Z^+ \cup \{0\}$) if and only if

$$\hat{\varphi}(\omega) = \hat{\varphi}_J^*(\omega) \hat{\varphi}(2^{-J}\omega) \tag{5}$$

holds, where $\hat{\varphi}_J^*(\omega) = \sum_n \hat{\varphi}(\omega + 2^{J+1} n\pi)$.

Following these works, Chen-Itoh [3] also established a general oversampling theory for wavelet subspace. Then we have the following structure

$$\begin{cases} \text{irregular} \\ \text{regular} \begin{cases} \text{sampling} \to \text{sampling property} \\ \text{oversampling} \to \text{oversampling property.} \end{cases} \end{cases}$$

However when we want to use (4) to recover signals from their discrete samples by scaling functions, two natural questions arise.

1. What kinds of scaling functions show oversampling property?

2. How to estimate the truncation error?

In this letter we will indicate that bounded interval band orthonormal scaling function shows oversampling property with rate $J \in Z^+$. We also present a estimation for the truncation error.

2. Bounded Interval Band Orthonormal Scaling Function Shows Oversampling Property

As the usual case we also study the orthonormal scaling function $\varphi(t)$ since any scaling function can be ortho-normalized (see Chui [6] and Mayer [10]), and we suppose that $|\varphi(t)| \leq O(|t|^{-1-\varepsilon})$ holds for some $\varepsilon > 0$ when $|t| \to \infty$. Firstly we need a proposition on bounded interval band scaling function which was given by Peng [11] (see also Walter [14]).

Proposition 1: Let $a, b \in R$ and $\mathrm{supp}\hat{\varphi}(\omega) = (a, b)$. Then

1. $a \le 0 \le b$, $2\pi \le b - a \le 8\pi/3$, $b/2 - a \le 2\pi$, and $b - a/2 \le 2\pi$.

2. $|\hat{\varphi}(\omega)| = 1$ for $\omega \in (a + \alpha, b - \alpha)$, where $\alpha = b - a - 2\pi$.

With the proposition we can now partially answer the first question in the Introduction.

Theorem 1: Let $a, b \in R$ and $\mathrm{supp}\hat{\varphi} = (a, b)$, Then $\varphi(t)$ shows oversampling property with rate J ($J \in Z^{+}$).

Proof: Since $\hat{\varphi}_J^*(\omega) = \sum_k \hat{\varphi}(\omega + 2^{J+1}k\pi)$, $b - a \le 8\pi/3$, and $J \in Z^{+}$, we have

$$\mathrm{supp}\hat{\varphi}_J^*(\omega) = \sum_k \{(a, b) + 2^{J+1}k\pi\}. \tag{6}$$

Following (1) of Proposition 1, we derive

$$2\pi + 2^{-J}a > b \quad \text{and} \quad a > 2^{-J}b - 2\pi, \tag{7}$$

i.e.,

$$2^{J+1}\pi + a > 2^J b \quad \text{and} \quad 2^J a > b - 2^{J+1}\pi. \tag{8}$$

Then (6) and (8) imply

$$\hat{\varphi}_J^*(\omega) = \hat{\varphi}(\omega) \quad \text{for} \quad \omega \in 2^J(a, b) = \mathrm{supp}\hat{\varphi}\left(\frac{\omega}{2^J}\right). \tag{9}$$

Let $\alpha = b - a - 2\pi$. Then

$$
\begin{aligned}
2^J(a + \alpha, &b - \alpha) \\
&= (2^J a + 2^J b - 2^J a - 2^{J+1}\pi, \\
&\quad 2^J b - 2^J b + 2^J a + 2^{J+1}\pi) \tag{10} \\
&= (2^J b - 2^{J+1}\pi, 2^J a + 2^{J+1}\pi) \supset (a, b), \tag{11}
\end{aligned}
$$

where (11) is due to (8). On the other hand, following (2) of Proposition 1 we derive

$$\hat{\varphi}\left(\frac{\omega}{2^J}\right) = 1 \quad \text{for} \quad \omega \in 2^J(a + \alpha, b - \alpha). \tag{12}$$

(9), (11) and (12) imply

$$
\begin{aligned}
\hat{\varphi}(\omega) &= \hat{\varphi}_J^*(\omega)\hat{\varphi}(\omega/2^J) \\
&\text{for} \quad \omega \in (a, b) = \mathrm{supp}\hat{\varphi}(\omega). \tag{13}
\end{aligned}
$$

Referring to the Introduction on oversampling property (see also Xia[15]) it is easy to see the conclusion from (13).

3. Truncation Error and an Example

When the scaling functions are applied to recover signals $f(t)$ from their discrete samples $\{f(\frac{n}{2^J})\}_n$, we must know how many items we should at least compute so that the recovered signal is as close to the original one as we expect. Then the truncation error defined by

$$T_f^e(t) = \sum_{|n| > N} f(2^{-J}n)\varphi(2^J t - n), \quad f \in V_0 \tag{14}$$

should be estimated. By a simple calculation we obtain the following estimation.

Theorem 2: Let $\varphi(t)$ be a scaling function of MRA $\{V_m\}_m$ with oversampling property of rate J. Then

$$
\left\| T_f^e(t) \right\| \le 2^{-J/2} \left(\sum_{|n| > N} \left| f(2^{-J}n) \right|^2 \right)^{1/2} \\
\cdot \left\| G_\varphi(\omega) \right\|_\infty, \tag{15}
$$

where $G_\varphi(\omega) = (\sum_k |\hat{\varphi}(\omega + 2k\pi)|^2)^{1/2}$, $\| \cdot \|$ denotes the $L^2(R)$-norm[†], and $\| \cdot \|_\infty$ denotes the essential superium[††].

For $f(t) \in V_m$, we know $f(2^{-m}t) \in V_0$. By (4) we have $f(2^{-m}t) = \sum_k f(2^{-m-J}k)\varphi(t - k)$. Hence

$$f(t) = \sum_k f(2^{-m-J}k)\varphi(2^m t - k),$$

$$\text{for} \quad f(t) \in V_m. \tag{16}$$

The truncation error is

$$
\left\| T_f^e(t) \right\| \le 2^{(-J-m)/2} \left(\sum_{|n| > N} \left| f(2^{-J-m}n) \right|^2 \right)^{1/2} \\
\cdot \left\| G_\varphi(\omega) \right\|_\infty. \tag{17}
$$

We now take the Meyer scaling function (see Walter[14]) as an example to show the above algorithms. Let scaling function $\varphi_M(t)$ be defined by

$$
\hat{\varphi}_M(\omega) \\
= \begin{cases}
1, & |\omega| \le \dfrac{2\pi}{3} \\[2mm]
\cos\left[\dfrac{\pi}{2}v\left(\dfrac{3}{2\pi}|\omega| - 1\right)\right], & \dfrac{2\pi}{3} \le |\omega| \le \dfrac{4\pi}{3} \\[2mm]
0, & \text{otherwise}
\end{cases}
$$

where $v(\omega) = 0$ when $\omega \le 0$, $v(\omega) = 1$ when $\omega \ge 1$ and $v(\omega) + v(1 - \omega) = 1$. Since $\varphi_M(t)$ is $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$ band, It must show oversampling property by Theorem 1. Hence we can use (16) to recover the signal $f(t) \in V_m$. Meanwhile the truncate error satisfies

$$\left\| T_f^e(t) \right\| \le 2^{\frac{-J-m}{2}} \left(\sum_{n > N} \left| f(2^{(-J-m)}n) \right|^2 \right)^{\frac{1}{2}} \tag{18}$$

due to $G_{\varphi_M}(\omega) = 1$.

[†] $\|f(t)\| = (\int_R |f(t)|^2 dt)^{1/2}$ for $f(t) \in L^2(R)$.

[††] $\|G_\varphi(\omega)\|_\infty = \inf_{|E|=0} \sup_{\omega \in R \ominus E} |G_\varphi(\omega)|$, where $|E|$ is the measure of the measurable subset E of R.

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