Wavelet Basis Packets and Wavelet Frame Packets

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ABSTRACT. This article obtains the nonseparable version of wavelet packets on $\mathbb{R}^d$ and generalizes the "unstability" result of nonorthogonal wavelet packets in Cohen–Daubechies to higher dimensional cases.

1. Introduction

The wavelet packets introduced by R. Coifman, Y. Meyer, and M. V. Wickerhauser played an important role in the applications of wavelet analysis as shown, for example, in [CMW1, CMW2]. But the theory itself is worthy of further study. Some developments in the wavelet packets theory should be mentioned, such as the tensor product version (due to [CM]) and the non-tensor-product version (due to [S]) of wavelet packets on $\mathbb{R}^d$, the nonorthogonal version of wavelet packets on $\mathbb{R}^1$ (due to [CL]), and the wavelet frame packets on $\mathbb{R}^1$ (due to [C]). The higher dimensional version of wavelet packets obtained in [S] is very close to the expected one. But it seems that there is a shortcoming in Shen's result; specifically, the implied frequency index is denoted by the point $\bar{n}$ in $\mathbb{Z}_+^d$, which makes the correspondence between the index pair $(\bar{n}, j)$ and the dyadic interval $I_{\bar{n},j}$ less natural than that in the one-dimensional cases. One task of this article is to set up a more natural framework for the wavelet packets in the higher dimensional case. Another task of this article is to study the lack of stability of nonorthogonal wavelet packets. As shown in [CD], starting from one-dimensional biorthogonal *multiresolution analysis* (MRA), a stable wavelet packet can hardly be constructed unless the matrix used in the splitting trick is unitary. We want to generalize the result to $\mathbb{R}^d$.

The notation and symbols used in this article are standard in wavelet theory. We list them as follows. For more detail see [LC].

An MRA is a nondecreasing family $\{V_j\}_{-\infty}^{\infty}$ of closed subspaces of $L^2(\mathbb{R}^d)$ satisfying:

i. $\bigcap V_j = \{0\}, \overline{\bigcup V_j} = L^2(\mathbb{R}^d)$;

ii. $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}, \forall j$;

iii. $\exists \varphi(x) \in V_0$ such that $\{\varphi(x - k)\}_k$ is a Riesz basis of V_0.

$\varphi(x)$ is called the scaling function of MRA $\{V_j\}_{-\infty}^{\infty}$, and $\varphi(x)$ satisfies the refinement equation $\exists \{d_k\} \in l^2$ such that

$$\varphi(x) = 2^d \sum_k d_k \varphi(2x - k) \text{ a.e. } x \in \mathbb{R}^d.$$

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The function

$$m_0(\xi) = \sum_k d_k e^{-ik\cdot\xi} \in L^2(T^d)$$

is called the filter function of $\{V_j\}_{-\infty}^{\infty}$. When the vector $(m_0(\xi + \nu\pi))_\nu$ ($\nu \in E_d = \{$all vertices of the cube $[0, 1]^d\}$) can be extended to a nonsingular matrix $M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}(\mu, \nu \in E_d)$ for a.e. ξ, with all $m_\mu(\xi)$ in $L^\infty(T^d)$, we can define the wavelet functions $\{\psi_\mu(x)\}_{\mu \in E_d}$ by

$$\hat{\psi}_\mu(2\xi) = m_\mu(\xi)\hat{\varphi}(\xi), \qquad \mu \in E_d \ (\varphi(x) = \psi_0(x)),$$

where the Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx \quad \forall f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

When the MRA $\{V_j\}_{-\infty}^{\infty}$ and $\{\tilde{V}_j\}_{-\infty}^{\infty}$ satisfy

$$\langle\varphi, \tilde{\varphi}(\cdot - k)\rangle = \int_{\mathbb{R}^d} \varphi(x)\overline{\tilde{\varphi}}(x - k)\,dx = \delta_{0,k},$$

we say that $\{V_j, \tilde{V}_j\}_{-\infty}^{\infty}$ is a biorthogonal MRA (pair) (in the case $\varphi = \tilde{\varphi}$, $\{V_j\}_{-\infty}^{\infty}$ is called an orthogonal MRA). Under some mild conditions, the following results have been established in [LC]. $\{V_j, \tilde{V}_j\}$ is biorthogonal if and only if $M(\xi)\overline{\tilde{M}'}(\xi) = I$ for a.e. ξ; when $\{V_j, \tilde{V}_j\}$ is biorthogonal and $M(\xi)$, $\tilde{M}(\xi)$ consist of entries in the class $C(T^d)$, then $\{\psi_{\mu,j,k}, \tilde{\psi}_{\mu,j,k}\}(\psi_{\mu,j,k}(x) = 2^{\frac{jd}{2}}\psi_\mu(2^j x - k))$, $j \in \mathbb{Z}, k \in \mathbb{Z}^d$, is biorthogonal in the sense

$$\langle\psi_{\mu,j,k}, \tilde{\psi}_{\mu',j',k'}\rangle = \delta_{\mu,\mu'}\delta_{j,j'}\delta_{k,k'}.$$

For $f, g \in L^2(\mathbb{R}^d)$, the bracket product of $\hat{f}$ and $\hat{g}$ is defined by

$$[\hat{f}, \hat{g}](\xi) = \sum_\alpha \hat{f}(\xi + 2\pi\alpha)\overline{\hat{g}}(\xi + 2\pi\alpha), \qquad \alpha \in \mathbb{Z}^d.$$

A sequence $\{e_j\}$ in a Hilbert space H is called a Riesz basis if $H = \overline{SP}(\{e_j\})$ ("$\overline{SP}$" means the closed, linear span) and

$$A\|\{c_j\}\|_2^2 \leq \left\|\sum_j c_j e_j\right\|^2 \leq B\|\{c_j\}\|_2^2 \quad \forall\{c_j\} \in l^2(\mathbb{Z})$$

is called a frame if

$$A\|f\|^2 \leq \sum_j |\langle f, e_j\rangle|^2 \leq B\|f\|^2 \quad \forall f \in H.$$

Notice that a Riesz basis $\{e_j\}$ is always a frame, and an independent frame is also a Riesz basis. When $\{e_j\}$ is a Riesz basis and a frame, then the Riesz basis bounds and the frame bounds are the same.

In what follows, we do not always start from an orthogonal MRA $\{V_j\}$ or a biorthogonal MRA $\{V_j, \tilde{V}j\}$, so the function φ or $\{\psi_\mu\}$ we treat need not be associated with some MRA $\{V_j\}$.

2. Orthogonal or Biorthogonal Wavelet Basis Packets and Wavelet Frame Packets

Just as Daubechies [D] indicated, the main tool in obtaining wavelet packets is the so-called splitting trick, which is a well-known technique in constructing wavelet bases. Since what we need is more general, we still state it as a lemma. The proof of the lemma will follow [LC].

Lemma 2.1.

Let $\varphi(x) \in L^2(\mathbb{R}^d)$ be such that $\{2^{\frac{d}{2}}\varphi(2x - k)\}_k$ is orthonormal. Denote $V = \overline{SP}(\{2^{\frac{d}{2}}\varphi(2x - k)\}_k)$. Let $\{u_{\mu,k}\}_k \in l^2(\mathbb{Z}^d)$, $\mu \in E_d$. Define

$$\psi_\mu(x) = 2^d \sum_k u_{\mu,k}\varphi(2x - k), \tag{2.1}$$

$$m_\mu(\xi) = \sum_k u_{\mu,k}e^{-ik\cdot\xi}, \qquad \xi \in T^d(= [0, 2\pi)^d = [-\pi, \pi)^d). \tag{2.2}$$

Then $\{\psi_\mu(x - k)\}_{\mu,k}$ is orthonormal if and only if $M(\xi) = (m_\mu(\xi + \nu\pi))$ $(\mu, \nu \in E_d)$ is a unitary matrix, for a.e. $\xi \in T^d$. Furthermore, $\{\psi_\mu(x - k)\}_{\mu,k}$ is an orthonormal basis of V whenever it is orthonormal.

Proof. We can get $\sum_\alpha |\hat{\varphi}(\xi + 2\pi\alpha)|^2 = 1$ a.e. ξ by the orthonormality of $\{2^{\frac{d}{2}}\varphi(2x - k)\}_k$. We have

$$\langle\psi_\mu, \psi_{\mu'}(\cdot - k)\rangle$$

$$= \int_{\mathbb{R}^d} \psi_\mu(x)\overline{\psi}_{\mu'}(x - k)\, dx$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} m_\mu\left(\frac{\xi}{2}\right)\overline{m_{\mu'}}\left(\frac{\xi}{2}\right)\left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 e^{ik\cdot\xi}\, d\xi \tag{2.3}$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{T^d} \sum_{\nu\in E_d} m_\mu\left(\frac{\xi}{2} + \nu\pi\right)\overline{m_{\mu'}}\left(\frac{\xi}{2} + \nu\pi\right) e^{ik\cdot\xi}\, d\xi.$$

Therefore

$$\langle\psi_\mu, \psi_{\mu'}(\cdot - k)\rangle = \delta_{\mu,\mu'}\delta_{o,k}$$

$$\Longleftrightarrow \sum_\nu m_\mu\left(\frac{\xi}{2} + \nu\pi\right)\overline{m_{\mu'}}\left(\frac{\xi}{2} + \nu\pi\right) = \delta_{\mu,\mu'} \quad \text{a.e. } \xi. \tag{2.4}$$

From (2.4) we see that $\{\psi_\mu(x - k)\}_{\mu,k}$ is orthonormal if and only if $M(\xi)$ is unitary for a.e. ξ. Now we assume that $\{\psi_\mu(x - k)\}_{\mu,k}$ is orthonormal and want to prove

$$\sum_{k\in\mathbb{Z}^d} \langle f, 2^{\frac{d}{2}}\varphi(2\cdot -k)\rangle 2^{\frac{d}{2}}\varphi(2x - k)$$

$$= \sum_{\mu\in E_d}\sum_{l\in\mathbb{Z}^d} \langle f, \psi_\mu(\cdot - l)\rangle\psi_\mu(x - l) \quad \forall f \in L^2(\mathbb{R}^d). \tag{2.5}$$

Once (2.5) is proved, $2^{\frac{d}{2}}\varphi(2x - k)$ can be expanded as a linear combination of $\{\psi_\mu(x - l)\}_{\mu,l}$, and hence $\{\psi_\mu(x - k)\}_{\mu,k}$ is an orthonormal basis of V.

Now we show (2.5). Since each side of (2.5) is L^2-convergent, in order to prove (2.5) it is enough to prove (2.5) in the weak sense, that is,

$$\sum_k \langle f, 2^{\frac{d}{2}} \varphi(2 \cdot -k) \rangle \langle g, 2^{\frac{d}{2}} \varphi(2 \cdot -k) \rangle^-$$

$$= \sum_\mu \sum_l \langle f, \psi_\mu(\cdot - l) \rangle \langle g, \psi_\mu(\cdot - l) \rangle^- \quad \forall f, g \in L^2. \tag{2.6}$$

Making use of Plancherel theorem and Parseval formula we have

$$I = \sum_\mu \sum_l \langle f, \psi_\mu(\cdot - l) \rangle \langle g, \psi_\mu(\cdot - l) \rangle^-$$

$$= \left(\frac{1}{2\pi} \right)^{2d} \sum_{\mu,l} \int_{T^d} \sum_\alpha \hat{f}(\xi + 2\pi\alpha) \overline{\hat{\psi}}_\mu(\xi + 2\pi\alpha) e^{il \cdot \xi} \, d\xi$$

$$\cdot \left(\int_{T^d} \sum_\beta \hat{g}(\xi + 2\pi\beta) \overline{\hat{\psi}}_\mu(\xi + 2\pi\beta) e^{il \cdot \xi} d\xi \right)^- \tag{2.7}$$

$$= \left(\frac{1}{2\pi} \right)^d \sum_\mu \int_{T^d} \sum_\alpha \hat{f}(\xi + 2\pi\alpha) \overline{\hat{\psi}}_\mu(\xi + 2\pi\alpha) \sum_\beta \overline{\hat{g}}(\xi + 2\pi\beta) \hat{\psi}_\mu(\xi + 2\pi\beta) \, d\xi.$$

Since from (2.1), (2.2) we have

$$\hat{\psi}_\mu(\xi) = m_\mu \left(\frac{\xi}{2} \right) \hat{\varphi} \left(\frac{\xi}{2} \right), \qquad \mu \in E_d, \tag{2.8}$$

substituting $\nu + 2\alpha'$ for α and $\nu' + 2\beta'$ for β in (2.7) and noticing the unitary property of $M(\xi)$ yields

$$I = \left(\frac{1}{2\pi} \right)^d \int_{T^d} \sum_\mu \sum_{\nu,\alpha'} \hat{f}(\xi + 2\pi\nu + 4\pi\alpha') \overline{m_\mu} \left(\frac{\xi}{2} + \nu\pi \right) \overline{\hat{\varphi}} \left(\frac{\xi}{2} + \nu\pi + 2\pi\alpha' \right)$$

$$\cdot \sum_{\nu',\beta'} \overline{\hat{g}}(\xi + 2\pi\nu' + 4\pi\beta') m_\mu \left(\frac{\xi}{2} + \nu'\pi \right) \hat{\varphi} \left(\frac{\xi}{2} + \nu'\pi + 2\pi\beta' \right) d\xi$$

$$= \left(\frac{1}{2\pi} \right)^d \int_{T^d} \sum_\nu \sum_{\alpha'\beta'} \hat{f}(\xi + 2\pi\nu + 4\pi\alpha') \overline{\hat{g}}(\xi + 2\pi\nu + 4\pi\beta')$$

$$\cdot \overline{\hat{\varphi}} \left(\frac{\xi}{2} + \nu\pi + 2\pi\alpha' \right) \hat{\varphi} \left(\frac{\xi}{2} + \nu\pi + 2\pi\beta' \right) d\xi \tag{2.9}$$

$$= \left(\frac{1}{2\pi} \right)^d \sum_{\nu \in E_d} \int_{[0,2\pi]^d + 2\pi\nu} \sum_{\alpha',\beta'} \hat{f}(\xi + 4\pi\alpha') \overline{\hat{g}}(\xi + 4\pi\beta') \overline{\hat{\varphi}} \left(\frac{\xi + 4\pi\alpha'}{2} \right) \hat{\varphi} \left(\frac{\xi + 4\pi\beta'}{2} \right) d\xi$$

$$= \left(\frac{1}{2\pi} \right)^d \int_{2T^d} \sum_\alpha \hat{f}(\xi + 4\pi\alpha) \overline{\hat{\varphi}} \left(\frac{\xi + 4\pi\alpha}{2} \right) \left(\sum_\beta \hat{g}(\xi + 4\pi\beta) \overline{\hat{\varphi}} \left(\frac{\xi + 4\pi\beta}{2} \right) \right)^- d\xi.$$

On the other hand, we have

$$\sum_k \langle f, 2^{\frac{d}{2}}\varphi(2\cdot-k)\rangle\overline{\langle g, 2^{\frac{d}{2}}\varphi(2\cdot-k)\rangle}^-$$

$$= \left(\frac{1}{2\pi}\right)^{2d}\frac{1}{2^d}\sum_k\int_{\mathbb{R}^d}\hat{f}(\xi)\overline{\hat{\varphi}}\left(\frac{\xi}{2}\right)e^{ik\cdot\frac{\xi}{2}}\,d\xi\overline{\left(\int_{\mathbb{R}^d}\hat{g}(\xi)\overline{\hat{\varphi}}\left(\frac{\xi}{2}\right)e^{i\cdot k\cdot\frac{\xi}{2}}\,d\xi\right)}^-$$

$$= \left(\frac{1}{2\pi}\right)^{d}\left(\frac{1}{4\pi}\right)^{d}\sum_k\int_{2T^d}\sum_\alpha\hat{f}(\xi+4\pi\alpha)\overline{\hat{\varphi}}\left(\frac{\xi+4\pi\alpha}{2}\right)e^{ik\cdot\frac{\xi}{2}}\,d\xi$$

$$\cdot\overline{\left(\int_{2T^d}\sum_\beta\hat{g}(\xi+4\pi\beta)\overline{\hat{\varphi}}\left(\frac{\xi+4\pi\beta}{2}\right)e^{ik\cdot\frac{\xi}{2}}\,d\xi\right)}^-$$

$$= \left(\frac{1}{2\pi}\right)^{d}\int_{2T^d}\sum_\alpha\hat{f}(\xi+4\pi\alpha)\overline{\hat{\varphi}}\left(\frac{\xi+4\pi\alpha}{2}\right)\sum_\beta\overline{\hat{g}}(\xi+4\pi\beta)\hat{\varphi}\left(\frac{\xi+4\pi\beta}{2}\right)\,d\xi. \qquad (2.10)$$

Combining (2.9) and (2.10), (2.7) follows. $\square$

Remark. The function $\varphi(x)$ in Lemma 2.1 is not necessarily the scaling function of a MRA, and the matrix $M(\xi)$ used in the splitting trick has no relationship to $\varphi(x)$ either. Hence we get more freedom in performing the splitting trick in what follows. $\square$

Lemma 2.1 has a biorthogonal version as follows.

Lemma 2.2.

Let φ *(and* $\tilde{\varphi}$*)* $\in L^2(\mathbb{R}^d)$ *be such that* $\{\varphi(x-k)\}_k$ *(and* $\{\tilde{\varphi}(x-k)\}_k$*) is a Riesz basis of the closed* $\mathbb{Z}^d$*-translation invariant subspaces* V_0 *(and* $\tilde{V}_0$*) generated by it and* $\{\varphi(x-k),\tilde{\varphi}(x-k)\}_k$ *be biorthogonal. Let* $V = 2V_0$ *(and* $\tilde{V} = 2\tilde{V}_0$*). Suppose that* $m_\mu(\xi), \tilde{m}_\mu(\xi) \in L^\infty(T^d)$ *for every* $\mu \in E_d$ *and*

$$m_\mu(\xi) = \sum_k u_{\mu,k}e^{-ik\cdot\xi} \quad and \quad \tilde{m}_\mu(\xi) = \sum_k \tilde{u}_{\mu,k}e^{-ik\cdot\xi}, \qquad \mu \in E_d. \qquad (2.11)$$

Define

$$\psi_\mu(x) = 2^d\sum_k u_{\mu,k}\varphi(2x-k) \quad and \quad \tilde{\psi}_\mu(x) = 2^d\sum_k\tilde{u}_{\mu,k}\tilde{\varphi}(2x-k), \qquad \mu \in E_d. \qquad (2.12)$$

Then $\{\psi_\mu(x-k), \tilde{\psi}_\mu(x-k)\}_{\mu,k}$ *is biorthogonal if and only if the matrices* $M(\xi) = (m_\mu(\xi+\nu\pi))_{\mu,\nu}$ *and* $\tilde{M}(\xi) = (\tilde{m}_\nu(\xi+\nu\pi))_{\mu,\nu}$ *satisfy*

$$M(\xi)\overline{\tilde{M}}^t(\xi) = I \quad a.e.\ \xi \in T^d, \qquad (2.13)$$

where the superscript t *means the transpose (hence* $\overline{\tilde{M}}^t = \tilde{M}^*$ *with* $*$ *denoting the conjugate). Furthermore, we have the direct sum*

$$V = \bigoplus_\mu\overline{SP}(\{\psi_\mu(x-k)\}_k), \qquad \tilde{V} = \bigoplus_\mu\overline{SP}(\{\tilde{\psi}_\mu(x-k)\}_k) \qquad (2.14)$$

whenever $\{\psi_\mu(x-k), \tilde{\psi}_\mu(x-k)\}_{\mu,k}$ *is biorthogonal.*

The proof is almost the same and can be omitted.

Lemma 2.1 can be used to yield a general result on the decomposition of Hilbert spaces, which is due to Coifman–Meyer–Wickerhauser [CMW2].

Proposition 2.3.

Let $d \in \mathbb{Z}_+$ and $\{e_k\}$ be any orthonormal basis of a Hilbert space H. Assume that $\{u_{\mu,k}\}_k \in l^2(\mathbb{Z}^d)$, $\mu \in E_d$, and define

$$m_\mu(\xi) = \sum_k u_{\mu,k} e^{-ik\cdot\xi} \quad and \quad f_{\mu,k} = 2^{\frac{d}{2}} \sum_k u_{\mu,2k-l} e_l, \qquad \mu \in E_d, k, l \in \mathbb{Z}^d. \tag{2.15}$$

Then $\{f_{\mu,k}\}_{\mu,k}$ is orthonormal if and only if the matrix $M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}$ is unitary for a.e. ξ. Furthermore $\{f_{\mu,k}\}_{\mu,k}$ is an orthonormal basis of H whenever $\{f_{\mu,k}\}_{\mu,k}$ is orthonormal.

Proof.   Find a $\varphi(x) \in L^2(\mathbb{R}^d)$ such that $\{\varphi(x-k)\}_k$ is orthonormal and define $V = \overline{\mathrm{SP}}(\{2^{\frac{d}{2}}\varphi$ $(2x-k)\}_k)$. Make the correspondence between e_k and $2^{\frac{d}{2}}\varphi(2x-k)$. Making use of (2.1), we define $\{\psi_\mu\}$. Then $\{f_{\mu,k}\}$ and $\{\psi_\mu(x-k)\}$ have a one-to-one correspondence. Proposition 2.3 is now deduced by Lemma 2.1. $\square$

We now turn to the construction of orthogonal wavelet packets. Let $\varphi(\in L^2(\mathbb{R}^d))$ and $M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}$ ($2\pi\mathbb{Z}^d$-periodic bounded measurable functions matrix) be given. Assume that $\{2^{\frac{d}{2}}\varphi(2x-k)\}_k$ is an orthonormal basis of $V = \overline{\mathrm{SP}}(\{2^{\frac{d}{2}}\varphi(2x-k)\}_k)$.

Applying the splitting trick to V, we get

$$\psi_\mu(x) = 2^d \sum_k u_{\mu,k}\varphi(2x-k), \qquad \hat\psi_\mu(\xi) = m_\mu\left(\frac{\xi}{2}\right)\hat\varphi\left(\frac{\xi}{2}\right). \tag{2.16}$$

Once again we get

$$\psi_{\mu_1,\mu_2}(x) = (\psi_{\mu_2})_{\mu_1}(x) = 2^d \sum_k u_{\mu_1,k}\psi_{\mu_2}(2x-k),$$

$$\hat\psi_{\mu_1,\mu_2}(\xi) = m_{\mu_1}\left(\frac{\xi}{2}\right)\hat\psi_{\mu_2}\left(\frac{\xi}{2}\right) = m_{\mu_1}\left(\frac{\xi}{2}\right)m_{\mu_2}\left(\frac{\xi}{4}\right)\hat\varphi\left(\frac{\xi}{4}\right). \tag{2.17}$$

Continuing in this way, for $j \in \mathbb{Z}_+$, we can define $\psi_{\mu_1,\dots,\mu_j}(x)$.

Now we simplify the index. Consider the 2^d-adic expansion of positive integers. For $n \in \mathbb{Z}_+$, we have unique $\mu = (\mu_1, \dots, \mu_j)$ such that

$$n = \mu_1 + 2^d\mu_2 + \cdots + 2^{(j-1)d}\mu_j, \qquad j = 0, 1, 2, 3, \dots,$$

with μ_i running through $0, 1, 2, \dots, 2^d - 1$ $\forall i$. When we order the elements of E_d as $0, 1, 2, \dots, 2^d - 1$ in any way, we can write $\mu_i \in E_d$ $\forall i$.

Let Δ_j be the set of these j-tuple $\mu = (\mu_1, \dots, \mu_j)$ with length j, and denote $\Delta = \bigcup_{j=1}^\infty \Delta_j$. Notice that when $i \leq j$, Δ_i can be imbedded in Δ_j naturally, by considering $(\mu_1, \dots, \mu_i)$ as $(\mu_1, \dots, \mu_i, 0, \dots, 0)$. Now, we rewrite (2.16) and (2.17). For $\mu_1 \in [0, 1, \dots, 2^d - 1] = E_d$, write $w_{\mu_1}(x) = \psi_{\mu_1}(x)$. For $\mu_1, \mu_2 \in E_d$, since $2^d\mu_2 + \mu_1$ is correspondent to (μ_1, μ_2), we write $w_{2^d\mu_2+\mu_1}(x) = \psi_{\mu_1,\mu_2}(x)$.

As such, we have

$$w_{\mu_1}(x) = 2^d \sum_k u_{\mu_1,k} \varphi(2x - k),$$

$$w_{2^d \mu_2 + \mu_1}(x) = 2^d \sum_k u_{\mu_1,k} w_{\mu_2}(2x - k).$$

In general, when $n = \mu_2 + \cdots + 2^{(j-2)d}\mu_j$, let $w_n(x) = \psi_{\mu_2,\ldots,\mu_j}(x)$. Since $2^d n + \mu_1 = \mu_1 + 2^d \mu_2 + \cdots + 2^{(j-1)d}\mu_j$, we can write

$$w_{2^d n + \mu_1}(x) = \psi_{\mu_1 \cdots \mu_j(x)}. \tag{2.18}$$

Hence we can rewrite the repeated splitting as

$$w_{2^d n + \mu_1}(x) = 2^d \sum_k u_{\mu_1,k} w_n(2x - k), \qquad n \in \mathbb{Z}_+, \mu_1 \in E_d. \tag{2.19}$$

Now we can formulate the first and the most fundamental result on wavelet packets.

Theorem 2.4.

Suppose that $\varphi(x)$ is a scaling function of an orthogonal MRA $\{V_j\}_{-\infty}^{\infty}$ of $L^2(\mathbb{R}^d)$ and $2\pi\mathbb{Z}^d$-periodic measurable functions matrix $M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}$ is unitary for a.e. ξ. Then $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ defined in (2.19) makes $\{w_n(x - k)\}_{n,k}$ an orthonormal basis of $L^2(\mathbb{R}^d)$.

Proof. We use the notation $w_n(x)$ and $w_\mu(x)$, when $n = (\mu_1,\ldots,\mu_j) = \mu$, to denote the functions defined in (2.19). We want to prove that $\{w_\mu(x - k)\}$ $(k \in \mathbb{Z}^d, \mu \in \Delta_j)$ or $\{w_n(x - k)\}$ $(k \in \mathbb{Z}^d, 0 \le n < 2^{jd})$ is an orthonormal basis of V_j $(j \ge 1)$ by induction.

By Lemma 2.1, when $j = 1$ we know that $\{w_\mu(x - k)\}$ $(\mu \in E_d, k \in \mathbb{Z}^d)$ is an orthonormal basis of V_1. Suppose that we have proved the assertion for j, that is to say $\{w_n(x-k)\}$ $(0 \le n \le 2^{jd}, k \in \mathbb{Z}^d)$ is an orthonormal basis of V_j $(j \ge 1)$. Since $V_{j+1} = \{f(2x) : f \in V_j\}$, $\{2^{\frac{d}{2}} w_n(2x - k)\}$ $(0 \le n < 2^{jd}, k \in \mathbb{Z}^d)$ consists of an orthonormal basis of V_{j+1}. Now the formula (2.19) and Lemma 2.1 show that

$$\{w_{2^d n + \mu_1}(x - k)\}, \qquad 0 \le n < 2^{jd}; \; \mu_1 = 0, 1, 2, \ldots, 2^d - 1; \; k \in \mathbb{Z}^d,$$

is an orthonormal basis of V_{j+1} too. Since

$$\{2^d n + \mu_1 : 0 \le n < 2^{jd}, \mu_1 = 0, 1, \ldots, 2^d - 1\} = \{n : 0 \le n < 2^{(j+1)d}\}, \tag{2.20}$$

we conclude that

$$\{w_n(x - k) : 0 \le n < 2^{(j+1)d}, k \in \mathbb{Z}^d\} = \{w_\mu(x - k) : \mu \in \Delta_{j+1}, k \in \mathbb{Z}^d\}$$

forms an orthonormal basis of V_{j+1}. Since $\overline{\bigcup V_j} = L^2(\mathbb{R}^d)$, we conclude that $\{w_n(x - k)\}$ $(k \in \mathbb{Z}^d, n \in \mathbb{Z}_+)$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. $\square$

Now we introduce the wavelet packets as in the one-dimensional case.

Definition 2.5. The family $\{2^{\frac{jd}{2}} w_n(2^j x - k)\}$, $n, j \in \mathbb{Z}_+$, $k \in \mathbb{Z}^d$, is called a wavelet basis packet, where n is called the oscillation parameter, j the scaling parameter, and k the location parameter. $\square$

The main results on wavelet packets is to characterize the set S of index pair (n, j), which makes $\{2^{\frac{jd}{2}} w_n(2^j x - k)\}$, $(n, j) \in S$, $k \in \mathbb{Z}^d$, being an orthonormal basis of $L^2(\mathbb{R}^d)$. To the index pair $(n, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ we correspond the dyadic interval

$$I_{n,j} = \{l \in \mathbb{Z}_+ : 2^{jd}n \le l < 2^{jd}(n + 1)\}. \tag{2.21}$$

The main result can be formulated in the same way as in the one-dimensional case.

Theorem 2.6.

Suppose that the conditions in Theorem 2.4 are satisfied and $S \subset \mathbb{Z}_+ \times \mathbb{Z}_+$. Then $\{2^{\frac{jd}{2}} w_n(2^j x - k)\}_{(n,j) \in S, k \in \mathbb{Z}^d}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ if and only if $\{I_{n,j}\}_{(n,j) \in S}$ is a disjoint covering of $\mathbb{Z}_+$.

Proof. Let

$$U_n = \overline{SP}(\{w_n(x - k)\}_k), \qquad 2U_n = \overline{SP}(\{2^{\frac{d}{2}} w_n(2x - k)\}_k). \tag{2.22}$$

Then Theorem 2.4 and Lemma 2.1 tell us that the following orthogonal direct sum decomposition holds

$$L^2(\mathbb{R}^d) = \bigoplus_n U_n; \qquad 2U_n = \bigoplus_{\mu_1} U_{2^d n + \mu_1}, \qquad \mu_1 = 0, 1, \ldots, 2^d - 1. \tag{2.23}$$

We now claim

$$2^j U_n = \bigoplus_l U_l, \qquad 2^{jd}n \le l < 2^{jd}(n + 1), \qquad n, j, l \in \mathbb{Z}_+. \tag{2.24}$$

It can be proved by induction. The case $j = 1$ follows from (2.23). Now deduce the $j + 1$ case from the j case. In fact, we have

$$2^{j+1} U_n = 2(2^j U_n) = \bigoplus_l 2U_l = \bigoplus_l \bigoplus_{\mu_1} U_{2^d l + \mu_1} = \bigoplus_m U_m, \qquad l \in I_{n,j}, \mu_1 \in E_d, m \in I_{n,j+1}, \tag{2.25}$$

where $m \in I_{n,j+1}$ can be seen as follows. The set $\{2^d l + \mu_1 : 2^{jd}n \le l < 2^{jd}(n + 1), \mu_1 = 0, 1, 2, \ldots, 2^d - 1\}$ consists of $2^{(j+1)d}$ $(= 2^d(2^{jd}(n + 1) - 2^{jd}n))$ integers, which are between $2^{(j+1)d}n$ and $2^{(j+1)d}(n + 1) - 1$ $(= 2^d(2^{jd}(n + 1) - 1) + 2^d - 1)$ and different from each other. This set is nothing but $I_{n,j+1}$; (2.24) is thus proved. Finally we get

$$\bigoplus_{(n,j) \in S} \overline{SP}(\{2^{\frac{jd}{2}} w_n(2^j x - k)\}_k) = \bigoplus_{(n,j) \in S} \bigoplus_{l \in I_{n,j}} U_l. \tag{2.26}$$

Therefore, the left-hand side of (2.26) is an orthogonal direct sum decomposition of $L^2(\mathbb{R}^d)$ if and only if $\bigcup_{(n,j) \in S} I_{n,j}$ is a partition of $\mathbb{Z}_+$. $\square$

Remark. Let $l \in \mathbb{Z}_+$, $S = \{(n, j) : (n, j) \in ([0, 2^{ld}) \times \{0\}) \cup ([2^{ld}, 2^{(l+1)d}) \times \mathbb{Z}_+)\}$. Then $\{I_{n,j}\}((n, j) \in S)$ forms a disjoint covering of $\mathbb{Z}_+$. In fact, we have

$$\bigcup_{2^{ld} \le n < 2^{(l+1)d}} I_{n,j} = [2^{(l+j)d}, 2^{(l+j+1)d}) \cap \mathbb{Z}_+, \qquad \bigcup_{(n,j) \in S} I_{n,j} = \mathbb{Z}_+.$$

Hence $\{2^{\frac{jd}{2}} w_n(2^j x - k) : (n, j) \in S, k \in \mathbb{Z}^d\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. In particular, when $l = 0$, this is exactly $\{w_0(x - k), 2^{\frac{jd}{2}} w_n(2^j x - k), 1 \le n \le 2^d - 1, j = 0, 1, \ldots, k \in \mathbb{Z}^d\}$,

which, when $m_0(\xi)$ is the filter function of an MRA, is the known wavelet basis. Another typical example of wavelet basis packets is that corresponding to $S = \mathbb{Z}_+ \times \{0\}$, that is,

$$\{w_n(x - k)\}, \qquad n \in \mathbb{Z}_+, k \in \mathbb{Z}^d.$$

The biorthogonal case is similar to the orthogonal case, modulo the stability. That is to say, if we want to get stable wavelet basis packets, in general, we can perform splitting operations only finitely many times. We will discuss this problem in detail in the next section. Here we give only some parallel results in the biorthogonal case. $\square$

Theorem 2.7.

Let $\{V_j, \tilde{V}_j\}$ be a biorthogonal MRA; $\{m_\mu\}, \{\tilde{m}_\mu\}$ be defined by (2.11) satisfying (2.13); and $\{w_n\}, \{\tilde{w}_n\}$ be defined by (2.19). Then for $j \in \mathbb{Z}_+, \{w_\mu(x - k)\}$ and $\{\tilde{w}_\mu(x - k)\}$ ($\mu \in \Delta_j, k \in \mathbb{Z}^d$) are Riesz basis of V_j and of $\tilde{V}_j$, respectively, and

$$\langle w_\mu(\cdot - k), \tilde{w}_\nu(\cdot - l) \rangle = \delta_{\mu,\nu}\delta_{k,l}, \qquad \mu, \nu \in \Delta_j, k, l \in \mathbb{Z}^d. \tag{2.27}$$

The proof is almost unchanged, and can be omitted.

Now we discuss what kind of results we can get by performing the splitting trick to wavelet frames. Chen [C] studied the problem in the one-dimensional case and obtained Lemma 2.8 and some similar results in following Theorems 2.9 and 2.10 with a different, less simple, and less natural formulation.

Let $\Phi = \{\varphi^{(r)}\}$ be a family consisting of n functions in $L^2(\mathbb{R}^d)$ and $S(\Phi) = \overline{SP}(\{\varphi^{(r)}(x-k)\}_{r,k})$. Let $P(\xi) = (p_{r,s}(\xi))_{r,s}$ be an $n \times n$ matrix with $2\pi\mathbb{Z}^d$-periodic bounded measurable functions as entries.

Define

$$\hat{\psi}^{(r)}(\xi) = \sum_{s=1}^{n} p_{r,s}(\xi)\hat{\varphi}^{(s)}(\xi), \qquad r = 1, \ldots, n. \tag{2.28}$$

Suppose that $\{\varphi^{(r)}(x - k)\}$ is a frame of $S(\Phi)$ with the upper bound B and the lower bound A, we want to discuss whether $\{\psi^{(r)}(x - k)\}_{r,k}$ is still a frame of $S(\Phi)$, and what is the upper bound and the lower bound of $\{\psi^{(r)}(x - k)\}_{r,k}$ when it is the case.

Lemma 2.8.

Assume that

$$C_1 I \leq P^*(\xi)P(\xi) \leq C_2 I \qquad a.e. \ \xi \in T^d, \tag{2.29}$$

where I denotes the identity matrix. Then for all $f \in L^2(\mathbb{R}^d)$ we have

$$C_1 \sum_r \sum_k |\langle f, \varphi^{(r)}(\cdot - k)\rangle|^2 \leq \sum_r \sum_k |\langle f, \psi^{(r)}(\cdot - k)\rangle|^2 \leq C_2 \sum_r \sum_k |\langle f, \varphi^{(r)}(\cdot - k)\rangle|^2. \tag{2.30}$$

On the contrary, when $\{\varphi^{(r)}(x - k)\}_{r,k}$ is a Riesz basis of $S(\Phi)$, then (2.29) is necessary for (2.30).

For the proof we refer to [C].

Now we apply the splitting trick to wavelet frames. Let $\varphi(x) \in L^2(\mathbb{R}^d)$ be such that $\{\varphi(x-k)\}_k$ is a frame of the space $V = \overline{SP}(\{\varphi(x - k)\}_k)$, and let $M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}$ be a nonsingular matrix for a.e. ξ where

$$m_\mu(\xi) = \sum_k u_{\mu,k}e^{-ik\cdot\xi} \in L^\infty(T^d), \qquad \mu \in E_d.$$

Define $\psi_\mu(x)$ as in (2.12), $\mu \in E_d$. Let

$$\varphi_\nu(x) = 2^{\frac{d}{2}}\varphi(2x - \nu), \qquad \nu \in E_d.$$

Then $\psi_\mu(x)$ has the equivalent expression

$$\psi_\mu(x) = \sum_{\nu,l} v_{\mu,\nu,l}\varphi_\nu(x - l), \quad \hat{\psi}_\mu(\xi) = \sum_\nu p_{\mu,\nu}(\xi)\hat{\varphi}_\nu(\xi), \qquad \mu = E_d, \tag{2.31}$$

where

$$v_{\mu,\nu,l} = 2^{\frac{d}{2}}u_{\mu,k}, \quad \text{when } k = 2l + \nu, \ l \in \mathbb{Z}^d, \ \mu, \nu \in E_d; \tag{2.32}$$

$$p_{\mu,\nu}(\xi) = \sum_l v_{\mu,\nu,l}e^{-il\cdot\xi}, \qquad \mu, \nu \in E_d. \tag{2.33}$$

The matrices

$$P(\xi) = (p_{\mu,\nu}(\xi))_{\mu,\nu} \quad \text{and} \quad M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}$$

are often used in the construction of wavelet bases. They obey the relationship (see [LC])

$$M\left(\frac{\xi}{2}\right) = P(\xi)2^{-\frac{d}{2}}\varepsilon\left(\frac{\xi}{2}\right), \qquad \varepsilon(\xi) = (\varepsilon_{\nu',\nu}(\xi))_{\nu',\nu}, \qquad \varepsilon_{\nu',\nu}(\xi) = e^{-i\nu'\cdot(\xi+\nu\pi)}. \tag{2.34}$$

Since $2^{-\frac{d}{2}}\varepsilon(\frac{\xi}{2})$ is unitary for every ξ, from

$$M^*\left(\frac{\xi}{2}\right)M\left(\frac{\xi}{2}\right) = 2^{-\frac{d}{2}}\varepsilon^*\left(\frac{\xi}{2}\right)P^*(\xi)P(\xi)2^{-\frac{d}{2}}\varepsilon\left(\frac{\xi}{2}\right)$$

we know that $M^*(\frac{\xi}{2})M(\frac{\xi}{2})$ and $P^*(\xi)P(\xi)$ are similar matrices and

$$C_1I \leq M^*\left(\frac{\xi}{2}\right)M\left(\frac{\xi}{2}\right) \leq C_2I \iff C_1I \leq P^*(\xi)P(\xi) \leq C_2I \quad \forall \xi. \tag{2.35}$$

Let $\Lambda(\xi)$ and $\lambda(\xi)$ be the maximal and minimal eigenvalues of the positive definite matrix $M^*(\xi)$ and $M(\xi)$, respectively; and let $\lambda = \inf_\xi \lambda(\xi)$ and $\Lambda = \sup_\xi \Lambda(\xi)$. When $0 < \lambda \leq \Lambda < \infty$, we know from Lemma 2.8 that

$$\lambda \sum_k |\langle f, 2^{\frac{d}{2}}\varphi(2 \cdot -k)\rangle|^2 \leq \sum_{\mu_1} \sum_k |\langle f, \psi_{\mu_1}(\cdot - k)\rangle|^2 \leq \Lambda \sum_k |\langle f, 2^{\frac{d}{2}}\varphi(2 \cdot -k)\rangle|^2, \tag{2.36}$$

where we have used the fact that

$$\sum_{\nu,l} |\langle f, \varphi_\nu(\cdot - l)\rangle|^2 = \sum_k |\langle f, 2^{\frac{d}{2}}\varphi(2 \cdot -k)\rangle|^2. \tag{2.37}$$

Performing the splitting trick to each ψ_{μ_1}, we get

$$\lambda \sum_k |\langle f, 2^{\frac{d}{2}}\psi_{\mu_1}(2 \cdot -k)\rangle|^2 \leq \sum_{\mu_2} \sum_k |\langle f, \psi_{\mu_2,\mu_1}(\cdot - k)\rangle|^2 \leq \Lambda \sum_k |\langle f, 2^{\frac{d}{2}}\psi_{\mu_1}(2 \cdot -k)\rangle|^2. \tag{2.38}$$

From (2.37), (2.38), and an induction argument, we see that for every $f \in L^2(\mathbb{R}^d)$ and $j \in \mathbb{Z}_+$, we have

$$\lambda^j \sum_k |\langle f, 2^{\frac{jd}{2}} \varphi(2^j \cdot -k)\rangle|^2 \leq \sum_{\mu_1,\ldots,\mu_j} \sum_k |\langle f, \psi_{\mu_1,\ldots,\mu_j}(\cdot -k)\rangle|^2 \leq \Lambda^j \sum_k |\langle f, 2^{\frac{jd}{2}} \varphi(2^j \cdot -k)\rangle|^2.$$
(2.39)

The arguments can be formulated to a theorem.

Theorem 2.9.

Let $\varphi(x) \in L^2(\mathbb{R}^d)$, $V_0 = \overline{SP}(\{\varphi(x - k)\}_k)$, and $\{\varphi(x - k)\}_k$ be a frame of V_0 with the upper bound B and the lower bound A. Assume that $M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}$ is a matrix of $2\pi\mathbb{Z}^d$-periodic bounded measurable functions satisfying $0 < \lambda \leq \Lambda < \infty$. Let $\{w_\mu(x)\} = \{w_n(x)\}$ be defined by (2.19). Then for all $j \in \mathbb{Z}_+$, $\{w_\mu(x - k)\}$, $\mu \in \Delta_j$, $k \in \mathbb{Z}^d$, is a frame of $V_j = \{f : f(2^{-j}) \in V_0\}$ with the upper bound $\Lambda^j B$ and the lower bound $\lambda^j A$.

Proof. Since $\{\varphi(x - k)\}_k$ is a frame of V_0 with the upper bound B and the lower bound A, we know that $\{2^{\frac{jd}{2}} \varphi(2^j x - k)\}_k$ is a frame of V_j with the same bounds for all j. By (2.39) and (2.18), we have

$$\lambda^j A \|f\|_2^2 \leq \sum_{\mu \in \Delta_j} \sum_k |\langle f, w_\mu(\cdot -k)\rangle|^2 \leq \Lambda^j B \|f\|_2^2 \quad \forall f \in V_j. \qquad \square$$
(2.40)

When $M(\xi)$ is unitary for a.e. ξ, the splitting trick can be operated for infinitely many times, as shown by the following theorem.

Theorem 2.10.

Let $\varphi \in L^2(\mathbb{R}^d)$ be such that $\{\varphi(x - k)\}_k$ is a frame of the space V_0 generated by itself with the bounds A and B, and let $V_0 \subset 2V_0$. Assume that $M(\xi)$ is unitary for a.e. ξ, then $\{w_n(x - k)\}$, $n \in \mathbb{Z}_+$, $k \in \mathbb{Z}^d$, is a frame of $(L^2(\Omega))^\vee$ with the same bound A and B, where

$$\Omega = \bigcup_j 2^j \text{ supp } \hat{\varphi}.$$
(2.41)

More generally, let $S = \{(n, j)\} \in \mathbb{Z}_+ \times \mathbb{Z}_+\}$ be such that $\bigcup_{(n,j)\in S} I_{n,j}$ is a partition of $\mathbb{Z}_+$; then $\{2^{\frac{jd}{2}} w_n(2^j x - k)\}_{(n,j)\in S, k\in\mathbb{Z}^d}$ is a frame of $(L^2(\Omega))^\vee$ with the same bounds A and B.

Proof. Since $\lambda = \Lambda = 1$, (2.39) becomes an equality and (2.40) becomes

$$A \|f\|_2^2 \leq \sum_{n=0}^{2^{jd}-1} \sum_k |\langle f, w_n(\cdot -k)\rangle|^2 \leq B \|f\|_2^2 \quad \forall f \in V_j.$$
(2.42)

By a result in [BDR], that is, $(\bigcup_j V_j)^- = (L^2(\Omega))^\vee$, we know that for any $f \in (L^2(\Omega))^\vee$ there exists a sequence $\{f_j\}$ such that $f_j \in V_j$ and $\lim_{j\to\infty} f_j = f$. We fix j at first; when $j \leq J$, we have

$$\sum_{n=0}^{2^{jd}-1} \sum_k |\langle f_J, w_n(\cdot -k)\rangle|^2 \leq B \|f_J\|_2^2.$$

Letting $J \to \infty$ at first and then $j \to \infty$, we get

$$\sum_{n=0}^{\infty} \sum_k |\langle f, w_n(\cdot -k)\rangle|^2 \leq B \|f\|_2^2 \quad \forall f \in (L^2(\Omega))^\vee.$$

Meanwhile we have

$$\|f_J\|^2 \le A^{-\frac{1}{2}} \left(\sum_{n=0}^{2^{Jd}-1} \sum_k |\langle f_J - f, w_n(\cdot - k)\rangle|^2 \right)^{\frac{1}{2}} + A^{-\frac{1}{2}} \left(\sum_{n=0}^{2^{Jd}-1} \sum_k |\langle f, w_n(\cdot - k)\rangle|^2 \right)$$

$$\le A^{-\frac{1}{2}} B^{\frac{1}{2}} \|f_J - f\|_2 + A^{-\frac{1}{2}} \left(\sum_{n=0}^{\infty} \sum_k |\langle f, w_n(\cdot - k)\rangle|^2 \right)^{\frac{1}{2}}.$$

Letting $J \to \infty$ (we get finally)

$$A\|f\|_2^2 \le \sum_0^{\infty} \sum_k |\langle f, w_n(\cdot - k)\rangle|^2.$$

The first assertion of the theorem has been proved.

Now we consider the general case. Assume that $S = \{(n, j) : \bigcup_{(n,j)\in S} I_{n,j}$ is a partition of $\mathbb{Z}_+\}$. Making use of Lemma 2.2 and the argument in the proof of Theorem 2.6, we know that the space generated by $\{2^{\frac{jd}{2}} w_n(2^j x - k)\}$ is $2^j U_n = \bigoplus_l U_l, l \in I_{n,j}$, where $\bigoplus$ denotes the direct sum (not necessarily orthogonal). In addition, owing to the equality (2.39) (in the case $\lambda = A$, (2.39) becomes an equality) we have

$$\sum_{\mu_1=0}^{2^d-1} \sum_k |\langle f, w_{2^d+\mu_1}(\cdot - k)\rangle|^2 = \sum_k |\langle f, 2^{\frac{d}{2}} w_n(2 \cdot -k)\rangle|^2,$$

$$\sum_{\mu_1,\mu_2} \sum_k |\langle f, w_{2^d(2^d n+\mu_1)+\mu_2}(\cdot - k)\rangle|^2 = \sum_k |\langle f, 2^{\frac{2d}{2}} w_n(2^2 \cdot -k)\rangle|^2.$$

For $j \in \mathbb{Z}_+$, the subscript of w in the left-hand side is $2^{jd} n + 2^{(j-1)d} \mu_1 + \cdots + \mu_j$. Since the set $\{2^{(j-1)d} \mu_1 + \cdots + \mu_j\} = \{0, 1, \ldots, 2^{jd} - 1\}$, the subscript of w runs through all integers from $2^{jd} n$ to $2^{jd}(n + 1) - 1$, that is, the integers in $I_{n,j}$.

Up to now, we have not only the direct sum decomposition $2^j U_n = \bigoplus_{l\in I_{n,j}} U_l$ but also the identity

$$\sum_k |\langle f, 2^{\frac{jd}{2}} w_n(2^j \cdot -k)\rangle|^2 = \sum_{l\in I_{n,j}} \sum_k |\langle f, w_l(\cdot - k)\rangle|^2. \tag{2.43}$$

By appealing to the first assertion of the theorem, for all $f \in (L^2(\Omega))^{\vee}$ we have

$$A\|f\|_2^2 \le \sum_{(n,j)\in S} \sum_{l\in I_{n,j}} \sum_k |\langle f, w_l(\cdot - k)\rangle|^2 = \sum_{n=0}^{\infty} \sum_k |\langle f, w_n(\cdot - k)\rangle|^2 \le B\|f\|_2^2. \tag{2.44}$$

Combining (2.43) and (2.44) we get

$$A\|f\|_2^2 \le \sum_{(n,j)\in S} \sum_k |\langle f, 2^{\frac{jd}{2}} w_n(2^j \cdot -k)\rangle|^2 \le B\|f\|_2^2 \quad \forall f \in (L^2(\Omega))^{\vee}. \qquad \square \tag{2.45}$$

Remark. The results in Theorem 2.10 cannot be transfered to the Riesz bases case in general. That is to say, starting from $V_0 = \overline{\mathrm{SP}}(\{\varphi(x-k)\}_k)$, where $\{\varphi(x-k)\}_k$ is a Riesz basis of V_0 with A, B as its bounds, perfoming the splitting trick with a unitary matrix $M(\xi)$, we cannot get a stable wavelet packet in general but can only get a wavelet frame packet. The reason and the counterexample have

been showed in [LC], where it was showed that when the filter function $m_0(\xi)$ of MRA $\{V_j\}$ permits a unitary extension, then under very mild condition, the wavelet functions $\{\psi_\mu\}_{\mu \in E_d - \{0\}}$ make $\{\psi_{\mu,j,k}\}$ a tight frame of $L^2(\mathbb{R}^d)$ and not being an orthonormal basis (the case $d = 1$ is due to W. Lawton [L]). $\quad \square$

3. The Instability of Nonorthogonal Wavelet Packets

In this section, we discuss what kind of conditions should be imposed on $M(\xi)$ when we want to get a wavelet frame of $L^2(\mathbb{R}^d)$ from a nonorthogonal MRA. Our intention is to generalize the result in [CD] to the higher dimensional case. We only consider the biorthogonal cases. At first, we discuss the necessary conditions imposed on $M(\xi)$ when we assume $\|w_n\|_2 = O(1)$. Notice that any frame $\{e_j\}$ of Hilbert space H satisfies $\|e_j\| = O(1)$ always. This comes from

$$\|e_i\|^4 = |\langle e_i, e_i \rangle|^2 \le \sum_j |\langle e_i, e_j \rangle|^2 \le B\|e_i\|^2 \quad \forall i.$$

Hence the condition $\|w_n\|_2 = O(1)$ is weaker than the frame property of $\{w_n(x - k)\}_{n,k}$.

Theorem 3.1.

Let $\{V_j, \tilde{V}_j\}$ be a biorthogonal MRA and $\varphi(x), \tilde{\varphi}(x)$ be the associated scaling functions. Assume that $M(\xi) = (m_\mu(\xi + \nu\pi))_{\mu,\nu}$ and $\tilde{M}(\xi) = (\tilde{m}_\mu(\xi + \nu\pi))_{\mu,\nu}$ are two matrices of $2\pi\mathbb{Z}^d$-periodic bounded measurable functions satisfying $M(\xi)\overline{\tilde{M}}^t(\xi) = I$ for a.e. ξ, where $\{m_\mu\}$ and $\{\tilde{m}_\mu\}$ are defined by (2.11). Suppose that $\|w_n\|_2 = O(1) = \|\tilde{w}_n\|_2$, where $\{w_n\}, \{\tilde{w}_n\}$ are defined in (2.19). Then, both of M^*M and $\tilde{M}^* \tilde{M}$ are diagonal matrices. More precisely (only see M^*M), we have

$$M^*(\xi)M(\xi) = \mathrm{diag}(p(\xi), \ldots, p(\xi + \nu\pi), \ldots), \qquad p(\xi) = \sum_\mu |m_\mu(\xi)|^2. \tag{3.1}$$

Proof. By the definition of w_n, for $\mu = (\mu_1, \ldots, \mu_j)$ we have $\hat{w}_n(\xi) = \prod_{i=1}^{j} m_{\mu_i}(2^{-i}\xi)\hat{\varphi}(2^{-j}\xi)$. Hence,

$$\sum_{n=0}^{2^{dJ}-1} \|w_n\|_2^2 = \left(\frac{1}{2\pi}\right)^d \sum_{\mu \in \Delta_J} \int_{\mathbb{R}^d} \prod_{j=1}^{J} |m_{\mu_j}(2^{-j}\xi)|^2 |\hat{\varphi}(2^{-J}\xi)|^2 \, d\xi$$

$$= \left(\frac{1}{2\pi}\right)^d \sum_{\mu \in \Delta_J} \int_{2^J T^d} \prod_{j=1}^{J} |m_{\mu_j}(2^{-j}\xi)|^2 \sum_\alpha |\hat{\varphi}(2^{-J}\xi + 2\pi\alpha)|^2 \, d\xi$$

$$\ge A2^{dJ} \left(\frac{1}{2\pi}\right)^d \sum_{\mu \in \Delta_J} \int_{T^d} \prod_{j=1}^{J} |m_{\mu_j}(2^{J-j}\xi)|^2 \, d\xi \tag{3.2}$$

$$= A2^{dJ} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \sum_{\mu \in \Delta_J} \prod_{j=0}^{J-1} |m_{\mu_{j+1}}(2^j\xi)|^2 \, d\xi.$$

Here we have used the fact that $\{\varphi(x - k)\}_k$ is a Riesz basis of V_0 (with the lower bound A). Since

$$\sum_{\mu \in \Delta_J} \prod_{j=0}^{J-1} |m_{\mu_{j+1}}(2^j\xi)|^2 = \prod_{j=0}^{J-1} \sum_{\mu \in E_d} |m_\mu(2^j\xi)|^2 = \prod_{j=0}^{J-1} p(2^j\xi),$$

we have

$$\sum_{n=0}^{2^{Jd}-1} \|w_n\|_2^2 \geq A2^{dJ} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \prod_{j=0}^{J-1} p(2^j\xi)\,d\xi. \tag{3.3}$$

Since $\sum_{n=0}^{2^{dJ}-1} \|w_n\|_2^2 = O(2^{dJ})$, (3.3) implies

$$\int_{T^d} \log p(\xi)\,d\xi \leq 0. \tag{3.4}$$

Otherwise, by the Jensen's inequality for convex functions, there would be $\delta > 0$, such that

$$\log \left(\frac{1}{2\pi}\right)^d \int_{T^d} \prod_{j=0}^{J-1} p(2^j\xi)\,d\xi \geq \left(\frac{1}{2\pi}\right)^d \int_{T^d} \log \prod_{j=0}^{J-1} p(2^j\xi)\,d\xi$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{T^d} \sum_{j=0}^{J-1} \log p(2^j\xi)\,d\xi$$

$$= J\left(\frac{1}{2\pi}\right)^d \int_{T^d} \log p(\xi)\,d\xi = \delta J,$$

and hence, we would have

$$\sum_{n=0}^{2^{Jd}-1} \|w_n\|_2^2 \geq A2^{Jd}2^{J\delta}.$$

The contradiction implies (3.4). In the same manner, we have $\int_{T^d} \log \tilde{p}(\xi)\,d\xi \leq 0$; hence

$$\int_{T^d} \log p(\xi)\tilde{p}(\xi)\,d\xi \leq 0. \tag{3.5}$$

Since $M^*\tilde{M} = I$ a.e. ξ, we have

$$1 = \left|\sum_\mu \overline{m}_\mu(\xi)\tilde{m}_\mu(\xi)\right|^2 \leq p(\xi)\tilde{p}(\xi).$$

By (3.5), we get $p(\xi)\tilde{p}(\xi) = 1$ a.e. ξ and, hence,

$$\prod_\nu p(\xi + \nu\pi) \prod_\nu \tilde{p}(\xi + \nu\pi) = 1 \quad \text{a.e. } \xi. \tag{3.6}$$

We want to use the Hadamard's inequality, which say that for any square matrix $A = (a_{ij})$ we have

$$|\det A|^2 \leq \prod_j \sum_i |a_{ij}|^2$$

and that the inequality becomes an equality if and only if the column vectors are orthogonal to each other. Suppose that either $|\det M(\xi)|^2 < \prod_\nu p(\xi + \nu\pi)$ or $|\det \tilde{M}(\xi)|^2 < \prod_\nu \tilde{p}(\xi + \nu\pi)$ hold on some set of positive measure; then on this set it would hold that

$$1 = |\det M(\xi)|^2|\det \tilde{M}(\xi)|^2 < \prod_\nu p(\xi + \nu\pi)\tilde{p}(\xi + \nu\pi) = 1.$$

The contradiction implies that

$$|\det M(\xi)|^2 = \prod_\nu p(\xi + \nu\pi) \quad \text{and} \quad |\det \tilde{M}(\xi)|^2 = \prod_\nu \tilde{p}(\xi + \nu\pi) \qquad \text{a.e. } \xi.$$

Therefore the column vectors of $M(\xi)$ are orthogonal to each other. Similarly, for $\tilde{M}(\xi)$ we have the same assertion. Thus M^*M and $\tilde{M}^*\tilde{M}$ are both diagonal matrices. $\square$

When both $p(\xi)$ and $\tilde{p}(\xi)$ are trigonometric polynomials, we can get more. In this case both M and $\tilde{M}$ can be shown to be unitary. For this we need some property of trigonometric polynomials.

Proposition 3.2.

Let $p(\xi)$ and $q(\xi)$ be trigonometric polynomials defined on T^d such that $p(\xi)q(\xi) \equiv 1$. Then

$$p(\xi) = \alpha e^{ik_0 \cdot \xi} \quad \text{and} \quad q(\xi) = \alpha^{-1} e^{-ik_0 \cdot \xi} \qquad \text{with } \alpha \in \mathbb{C}, k_0 \in \mathbb{Z}^d. \tag{3.7}$$

Proof. First we consider the one-dimensional case. Let $p(\theta) = \sum_k \alpha_k e^{ik\theta}$, $q(\theta) = \sum_m \beta_m e^{im\theta}$, and $[K_1, K_2]$ and $[M_1, M_2]$ be the minimal supporting interval of the coefficients of $p(\xi)$ and $q(\xi)$, respectively. Since $0 \neq \alpha_{K_2}\beta_{M_2} = \delta_{K_2,-M_2}$, we get $K_2 = -M_2$, and $K_1 \leq K_2 = -M_2 \leq -M_1$. Similarly from $0 \neq \alpha_{K_1}\beta_{M_1} = \delta_{K_1,-M_1}$, we get $K_1 = -M_1$, and $K_1 = K_2 = -M_1 = -M_2$. This implies that $p(\theta) = \alpha e^{ik_0\theta}$ and $g(\theta) = \alpha^{-1} e^{-ik_0\theta}$. This proves the proposition in the one-dimensional case. Now consider the d-dimensional case. Let

$$p(\xi) = \sum_k \alpha_k(\xi') e^{ik\xi_d}, \qquad q(\xi) = \sum_m \beta_m(\xi') e^{im\xi_d},$$

where $\alpha_k(\xi')$ and $\beta_m(\xi')$ are trigonometric polynomials of $d - 1$ dimension. From $p(\xi)q(\xi) = 1$, we have

$$p(\xi) = \alpha(\xi') e^{ik_d\xi_d} \quad \text{and} \quad g(\xi) = \beta(\xi') e^{-ik_d\xi_d}, \qquad \alpha(\xi')\beta(\xi') = 1, \tag{3.8}$$

where $\alpha(\xi')$ and $\beta(\xi')$ are both trigonometric polynomials. Suppose that the assertion has been proved in the $(d - 1)$-dimensional case. Then

$$\alpha(\xi') = \alpha \prod_{j=1}^{d-1} e^{ik_j\xi_j}, \qquad \beta(\xi') = \alpha^{-1} \prod_{j=1}^{d-1} e^{-ik_j\xi_j}. \tag{3.9}$$

Thus we have

$$p(\xi) = \alpha \prod_{j=1}^{d} e^{ik_j\xi_j}, \qquad q(\xi) = \alpha^{-1} \prod_{j=1}^{d} e^{-ik_j\xi_j}. \quad \square$$

Theorem 3.3.

Let $\{V_j, \tilde{V}_j\}$, $\varphi(x)$, $\tilde{\varphi}(x)$, $M(\xi)$, $\tilde{M}(\xi)$, w_n, and $\tilde{w}_n$ be the same as in Theorem 3.1. In addition, assume that $p(\xi)$, $\tilde{p}(\xi)$ are both trigonometric polynomials. Then both of $M(\xi)$ and $\tilde{M}(\xi)$ are unitary a.e. ξ.

Proof. From Theorem 3.1, we know that the column vectors of $M(\xi)$ (and of $\tilde{M}(\xi)$) are orthogonal to each other and that $p(\xi)\tilde{p}(\xi) \equiv 1$. Using Proposition 3.2, we see that the nonnegative polynomials $p(\xi) = \alpha$, $\tilde{p}(\xi) = \alpha^{-1}$. But $p(0) = 1 = \tilde{p}(0)$, so we get $p(\xi) \equiv 1 \equiv \tilde{p}(\xi)$, which implies that $M^*M = I$; hence $M(= \tilde{M})$ is unitary. $\square$

Remark. The one-dimensional case can be found in [CD]. □

Finally, we discuss a related problem. Assume that $\{V_j, \tilde{V}_j\}$ is a biorthogonal MRA, $\varphi(x)$ and $\tilde{\varphi}(x)$ are the associated scaling functions, and $m_0(\xi)$ and $\tilde{m}_0(\xi)$ are the associated filter functions. Suppose that we have the matrices $M(\xi) = (m_\mu(\xi+\nu\pi))_{\mu,\nu}$ and $\tilde{M}(\xi) = (\tilde{m}_\mu(\xi+\nu\pi))_{\mu,\nu}$ satisfying $M(\xi)\tilde{M}^*(\xi) = I$ a.e. ξ and $m_\mu, \tilde{m}_\mu \in L^\infty(T^d)$, $\mu \in E_d$. We now perform the splitting trick using these two matrices. Suppose that both $\{w_n(x - k)\}_{n,k}$ and $\{\tilde{w}_n(x - k)\}_{n,k}$ (with $\{w_n\}$, $\{\tilde{w}_n\}$ defined by (2.19)) are frames of $L^2(\mathbb{R}^d)$ with bounds A, B and $\tilde{A}$, $\tilde{B}$, respectively. The question is what kind of estimates for bounds λ, Λ and $\tilde{\lambda}$, $\tilde{\Lambda}$ of the eigenvalues of M^*M and $\tilde{M}^*\tilde{M}$ can be obtained.

Proposition 3.4.

Let $\{V_j, \tilde{V}_j\}$ be a biorthogonal MRA and $\varphi(x)$, $\tilde{\varphi}(x)$ and $m_0(\xi)$, $\tilde{m}_0(\xi)$ be its scaling functions and filter functions, respectively. Assume that there is an extension $\{m_\mu, \tilde{m}_\mu\}_{\mu\in E_d}$ of $\{m_0, \tilde{m}_0\}$ satisfying

$$M(\xi)\tilde{M}^*(\xi) = I \quad a.e.\ \xi;$$
$$m_\mu, \tilde{m}_\mu \in L^\infty(T^d), \qquad \mu \in E_d.$$

Suppose that $\{w_n(x - k)\}_{n,k}$ and $\{\tilde{w}_n(x - k)\}_{n,k}$ (with $\{w_n\}$, $\{\tilde{w}_n\}$ defined by (2.19)) are both frames of $L^2(\mathbb{R}^d)$, with bounds A, B and $\tilde{A}, \tilde{B}$, respectively. Then $M^*(\xi)M(\xi)$ and $\tilde{M}^*(\xi)\tilde{M}(\xi)$ satisfy, respectively,

$$B^{-1}AI \leq M^*(\xi)M(\xi) \leq A^{-1}BI \quad and \quad \tilde{B}^{-1}\tilde{A}I \leq \tilde{M}^*(\xi)\tilde{M}(\xi) \leq \tilde{A}^{-1}\tilde{B}I, \qquad a.e.\ \xi.$$

Proof. We know that $\{w_\mu(x - k)\}$ and $\{\tilde{w}_\mu(x - k)\}$, $\mu \in \Delta_j$, $k \in \mathbb{Z}^d$, are Riesz bases of V_j and $\tilde{V}_j$, respectively, and that $\{w_\mu(x - k), \tilde{w}_\mu(x - k)\}_{\mu,k}$ is biorthogonal. Hence, we have

$$f(x) = \sum_{n=0}^{2^{jd}-1} \sum_k \langle f, \tilde{w}_n(\cdot - k)\rangle w_n(x - k) \quad \forall f \in V_j. \tag{3.10}$$

We will only consider the cases $j = 0, 1$. Notice that $w_0 = \varphi$, $\tilde{w}_0 = \tilde{\varphi}$ and $w_\mu = \psi_\mu$, $\tilde{w}_\mu = \tilde{\psi}_\mu$, $\mu \in E_d$. Since $\{w_n(x - k)\}_{n,k}$ is a frame of $L^2(\mathbb{R}^d)$ with upper bound B and lower bound A, for $j = 0, 1$ we have, respectively,

$$A\|f\|_2^2 \leq \sum_\mu \sum_k |\langle f, \psi_\mu(\cdot - k)\rangle|^2 \leq B\|f\|_2^2 \quad \forall f \in \tilde{V}_1, \tag{3.11}$$

$$A\|f\|_2^2 \leq \sum_k |\langle f, \varphi(\cdot - k)\rangle|^2 \leq B\|f\|_2^2 \quad \forall f \in \tilde{V}_0. \tag{3.12}$$

Then (3.12) can be rewritten as

$$A\|f\|_2^2 \leq \sum_k |\langle f, 2^{\frac{d}{2}}\varphi(2\cdot -k)\rangle|^2 \leq B\|f\|_2^2 \quad \forall f \in \tilde{V}_1. \tag{3.13}$$

Now we define operators

$$\tilde{P}_j f(x) = \sum_{n=0}^{2^{jd}-1} \sum_k \langle f, w_n(\cdot - k)\rangle \tilde{w}_n(x - k) \quad \forall f \in L^2(\mathbb{R}^d), \tag{3.14}$$

and its counterpart P_j. Reasoning just like in [LC], we see that $\tilde{P}_j$ is a projection from $L^2(\mathbb{R}^d)$ onto $\tilde{V}_j$ and the dual of P_j. Hence, (3.11) and (3.13) become

$$A\|\tilde{P}_1 f\|_2^2 \leq \sum_\mu \sum_k |\langle f, \psi_\mu(\cdot - k)\rangle|^2 \leq B\|\tilde{P}_1 f\|_2^2 \quad \forall f \in L^2(\mathbb{R}^d), \tag{3.15}$$

$$A\|\tilde{P}_1 f\|_2^2 \leq \sum_k |\langle f, 2^{\frac{d}{2}}\varphi(2 \cdot -k)\rangle|^2 \leq B\|\tilde{P}_1 f\|_2^2 \quad \forall f \in L^2(\mathbb{R}^d). \tag{3.16}$$

Here we have used the duality between P_1 and $\tilde{P}_1$ and the facts

$$P_1(\psi_\mu(\cdot - k)) = \psi_\mu(x - k), \qquad P_1(2^{\frac{d}{2}}\varphi(2 \cdot -k)) = 2^{\frac{d}{2}}\varphi(2x - k).$$

Therefore,

$$B^{-1}A \sum_k |\langle f, 2^{\frac{d}{2}}\varphi(2 \cdot -k)\rangle|^2 \leq \sum_\mu \sum_k |\langle f, \psi_\mu(\cdot - k)\rangle|^2$$

$$\leq A^{-1}B \sum_k |\langle f, 2^{\frac{d}{2}}\varphi(2 \cdot -k)\rangle|^2 \quad \forall f \in L^2. \tag{3.17}$$

By Lemma 2.8, (3.17) implies $B^{-1}AI \leq M^*M \leq A^{-1}BI$. The same argument works well for $\tilde{M}^*\tilde{M}$. $\square$

Remark. If one of $\{w_n(x - k)\}_{n,k}$ and $\{\tilde{w}_n(x - k)\}_{n,k}$ is a tight frame $(A = B)$, then $M(= \tilde{M})$ is unitary. $\square$

4. Conclusions

1. The natural indexes are introuduced for the splitting trick, which admit that the splitted results can be simply formulated as $\{\omega_n(x)\}_{n\in\mathbb{Z}_+}$ $(\{\tilde{\omega}_n(x)\}_{n\in\mathbb{Z}_+})$.

2. In orthogonal wavelet subspaces, a sufficient condition for $\{\omega_n(x)\}_{n\in\mathbb{Z}_+}$ to be an orthogonal wavelet basis packet of $L^2(\mathbb{R}^d)$ is that the $M(\xi)$ used to perform a splitting trick is unitary, and a sufficient-necessary condition is that $\{I_{(n,j)}\}_{(n,j)\in\mathbb{Z}_+\times\mathbb{Z}_+}$ is a disjoint cover of $\mathbb{Z}_+$.

3. The sufficient conditions for $\{\omega_n(x)\}_{n\in\mathbb{Z}_+}$ to be a wavelet frame packet of V_j are that $M(\xi)$ is positive definite and bounded and to be a wavelet frame packet of $L^2(\Omega)$ are that $M(\xi)$ is unitary or that $\{I_{(n,j)}\}_{(n,j)\in\mathbb{Z}_+\times\mathbb{Z}_+}$ is a partition of $\mathbb{Z}_+$ where $\Omega = \bigcup_j 2^j \text{supp } \hat{\varphi}$.

4. In biorthogonal wavelet subspaces, when $M(\xi)\tilde{M}^*(\xi) = I$ the necessary conditions for $\{\omega_n(x)\}_{n\in\mathbb{Z}_+}$ $(\{\tilde{\omega}_n(x)\}_{n\in\mathbb{Z}_+})$ to be a frame of $L^2(\mathbb{R})$ are that $M(\xi)M^*(\xi)$ $(\tilde{M}(\xi)\tilde{M}^*(\xi))$ is diagonal and satisfies

$$B^{-1}AI \leq M^*(\xi)M(\xi) \leq A^{-1}BI \text{ or } \tilde{B}^{-1}\tilde{A}I \leq \tilde{M}^*(\xi)\tilde{M}(\xi) \leq \tilde{A}^{-1}\tilde{B}I \quad \text{a.e. } \xi.$$

Furthermore, $M(\xi)$ $(\tilde{M}(\xi))$ needs to be unitary if $M(\xi)$ $(\tilde{M}(\xi))$ is of polynomial entries.

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